Key Words: Classical laminate theory, laminates, laminae, elasticity relationships

Summary

This HSB sheet presents how stresses and strains in a laminate are determined by the Classical Laminate Theory (2D-CLT, in-plane).
Many examples are included to demonstrate the application of this linear analysis theory. Principally, the procedure is not only valid for UD laminae. Also cross-ply textiles may be treated regarding the stiffness analysis.

The designations follow Ref. [3] and [4], but not the Poisson ratios. In the Annex, the different use of the layer numbering in design, in FE-codes and in manufacturing is addressed. Further conventions are locally used.

The obtained stresses are inserted into the strength failure conditions (criteria) to judge whether one of the two fiber failure modes (FF) or of the three inter-fiber failure modes (IFF) is met. Both these failure mode families are inherent to UD materials.

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Contents

1 General

This work sheet describes the analysis of laminates on basis of the 2D or in-plane continuum theory, respectively. Usually, 2D continuum theory applied to laminates is referred to as Classical Laminate Theory (CLT). It includes the so-called Kirchhoffian plate theory and is used to determine the in-plane stresses and in-plane strains in each individual lamina (ply) of the laminate.

Basic references are Ref. [1] and [2]. An extended CLT theory is the content of Ref. [14]

The denotations follow Ref. [3], for definitions see Ref. [4] and [7].
A layer is a physical sub-unit of the laminate, whereas a lamina (ply) is a numerical building block in the laminate analysis which might be half of a layer or also a single layer of a non-crimped fabric. In this context four specifica are regarded:

1. The coordinate system with the reference plane
2. The fiber orientation angle α
3. The lay-up with the laminate stacking sequence and
4. The layer numbering or counting sequence.
   In the respective figures all these specifica are pointed out.

In literature, the direction of the coordinate system is arbitrarily chosen, however, right hand system is applied. For instance in Fig. 2 and Fig. 3, this is done differently to optimally visualize the respective contents addressed in each single figure.

In order to avoid any confusion all the examples in chapter 4 will follow the definitions in Fig. 3.

2 List of Symbols

Symbol	Unit	Description
k	–	number of the single lamina (ply)
m_{xy}	N·mm/mm	laminate's torsion (twisting) section moment
m_x, m_y	N·mm/mm	laminate's bending section moments = bending stress resultant per unit width
n^0_{xy}	N/mm	section shear force
n^0_x, n^0_y	N/mm	section normal force = membr. stress resultant p. u. width
n	–	total number of laminae
t, t_k	mm	laminate (wall) thickness, thickness of the k^{th}lamina (CDM-!7: h = t/2; Tsai : h = t)
x, y, z	mm	coordinates of laminate (and also of the structural part)
x_1, x_2, x_3	mm	coordinates of lamina (1= ‖),
r	–	number of boundary conditions (Fig. 6)
u, v, w	mm	displacements of laminate element in the reference plane
z_{off_x}, z_{off_y}	mm	distance of the action plane of n^0_x, n^0_y (analogous n^0_{xy})
$[A]$, $\boldsymbol{A}$	N/mm	extensional stiffness matrix of laminate (in-plane stretching), A_{ij}= elements of the laminate's membrane (extensional) stiffness matrix or membrane stiffnesses
$[A^*]$	mm/N	inverted extensional stiffness matrix (part of $[K]^{-1}$)
$[B]$, $\boldsymbol{B}$	N N	coupling stiffness matrix of laminate (bending-stretching), B_{ij}= elements of the laminate's coupling stiffness matrix or coupling stiffnesses (values depend on reference plane)
$[B^*]$	1/N	inverted coupling stiffness matrix (part of $[K]^{-1}$)
$[C]$, $\boldsymbol{C}$	MPa	3D elasticity matrix of UD lamina material in lamina COS, C_{ij}= elements of the lamina'a (ply's) elasticity matrix or elasticity stiffnesses (sometimes termed stiffn. coefficients)
$[C']$	MPa	3D elasticity matrix of UD lamina material in laminate COS
$[D]$, $\boldsymbol{D}$	N·mm $\equiv$ MPa · mm^3	bending stiffness matrix of laminate (plate stiffness matrix), D_{ij}= elements of the laminate's bending (flexural) stiffness matrix or bending stiffnesses or flexural rigidities
$[D^*]$	$1/\mathrm{N}\cdot\mathrm{mm}$	inverted bending stiffness matrix of laminate, part of $[K]^{-1}$
E	MPa	elasticity modulus (Young's modulus), from uniaxial testing
$\widehat{E}, \widehat{G}$	MPa	elasticity 'constants' of the laminate
$\widehat{EA}$	N/mm or MPa · mm	effective axial stiffness per unit width of homogen. laminate
$\widehat{EI}$	N · mm or MPa · mm^3	eff. bending stiffness per unit width of homogenized plate
EI	N · mm^2	bending stiffness of the beam $b \cdot \widehat{EI}$ (isotropic: $EI = E \cdot I$)
$\widehat{GI_t}$	N · mm	effective torsional stiff. per unit width of homogen. laminate
F	N/mm N·mm/mm	vector of section forces $\{n^0\}$ in Fig. 6 and of section moments $\{m\}$
$[K]$	$^{(a)}$	laminate stiffness matrix, K_{ij}= elements of the laminate stiffness matrix (stiffnesses)

$[K]^{-1}$	(a)	laminate compliance matrix
M_t	$\mathrm{N \cdot mm}$	torque or twisting moment of the beam
M_y, M_{xy}	$\mathrm{N \cdot mm}$	bending moment, torsion moment of the beam
$[Q]$	MPa	in-plane (2D) elasticity matrix of UD lamina in lamina COS
$[Q']$	MPa	in-plane (2D) elasticity matrix of UD lam. in laminate COS
$[S]$	1/MPa	3D compliance matrix of UD lamina mater. in lamina COS
$[S']$	1/MPa	3D compliance matrix of UD lamina mat. in laminate COS
$[T_\sigma]$, $[T_\varepsilon]$	–	transformation matrices for the vectors $\{\sigma\}$, $\{\varepsilon\}$
V_f	–	fiber volume ratio (in practice often given in %); capital V is generally chosen, because a small italic v looks like a nue ν)
α	°	orientation angle of lamina, here measured from x to x_1
α_M	$10^{-4} \cdot \mathrm{mm/(mm \cdot \%)}$	coefficient of moisture expansion CME (moisture change) (in literature also is used: $\alpha_M = \beta$, $\alpha_T = \alpha$
α_T	$10^{-6} \cdot \mathrm{mm/(mm \cdot K)}$	coefficient of thermal expansion CTE (temperature change)
ε	%	normal strain (usually provided in %, sometimes in 10^{-3}mm/mm = 10^{-1}%= $1^{o}/_{oo}$ or in microstrain μm/m = 10^{-6}mm/mm = 10^{-4} %)
γ	%	shear strain (in practice, property data are provided in %)
$\{\varepsilon^0\}$	mm/mm	strain vector of the laminate's reference plane
$\{\delta\}$	mm	vector of displacements
$\{\kappa\}$	1/mm	curvature vector of the laminate's reference plane
κ_{xy}	1/mm	twisting curvature (definition in CMH-17 different)
ν	–	Poisson's ratio (suffixes: see Annex)
σ, τ	MPa	normal stress, shear stress
ϑ'	rad/mm	specific angle of twist (related to x-axis) = derivative of ϑ
φ	rad	bending angle
ϑ	rad	angle of twist $\vartheta = L \cdot \vartheta'$
ω	°	half crossing angle of angle-ply lam. ($\alpha_1 = \omega, \alpha_2 = -\omega$)

[a] same unit as property under consideration

Abbreviation	Description
CME	coefficient of moisture expansion
COS	coordinate system
CTE	coefficient of thermal expansion
ELT	extended laminate theory
FF, IFF	fiber failure, inter-fiber-failure
FRP	fiber reinforced plastics
LSS	laminate stacking sequence

	Laminate Conventions
	C = carbon, G = glass, A = aramide, M = mat, W = warp, F = fill (weft)
distinct laminate symm. stack	UD stack or lay-up of laminae (plies) $\left[0_2^C/45^G/-45^G/\underline{90}^{GM}\right]_S \equiv$ $\equiv[0^C/0^C/45^G/-45^G/90^{GM}/-45^G/45^G/0^C/0^C]$, $_S = 2$
repeated sub-laminates	sandwich (core with face sheets, $_r$= repeat) $[[0/45/\text{-}45]_r/\text{core}]_S \equiv$ $\equiv[[0/45/-45/0/45/-45/\text{core}/-45/45/0/-45/45/0]$
laminate family ply contributions	(0°, ±45°, 90°) example: (20,70,10) ≡ (20% 0°, 70%±45°, 10% 90°)
laminae (plies)	transversely-isotropic UD lamina, rhombically anisotropic WF lamina (fabric), transv.-isotropic Mat lamina (is in-plane quasi-isotropic)
lay-up	defines the laminate family (due to CMH-17)
stack	fixed laminate stacking sequence (optimized to reach minimum coupling)

Indices	**Description**
$_0$	reference plane $z = 0$ (the mid-plane is often used as reference plane)
$_{1,2,3}$	numeric subscripts designate the coordinate axes of the lamina (1 = fiber direct. ∥). Note: In CMH-17 and with Tsai (for instance) $_{1,2,3}$ corresponds to $_{x,y,z}$!
$_{ef}$	effective
W, F	warp, fill of an orthotropic fabric (modelled by two UD laminae
$_{i,j}$	indices with numbers 1,2, 6
$_{i,k}$	running indices i,k
$_L$	loading
$_r$	reduced equation system (Fig. 6), repeat index in lay-ups
$_{S}$, $_{-}$	symmetric, half lamina thickness
$_{x,y,z}$	letter subscripts, designating the coordinate axes of the laminate (usually used as structural coordinates, too)
$_{xy}$	marks the twisting section moment
$_t$	torsion
$''$	second derivation of a displacement (e.g. $w'' = d^2w/dx^2$)
$^-$	average (mean, if Gauss distribution) marking sign (standard sign in statistics)
$_{\perp,\parallel}$	symbolic designation of lamina quantities (perpendicular, parallel)
$'$	superscript for a stress or strain (σ', ε') in the laminate COS (some books like MiL-HDBK-17 III, now CMH-17 (vol. 3), use the average sign$^-$, instead)
*	asterix superscript designation for inverted matrix elements
$\breve{}$	superscript designation of a 'mixed' quantity
$\widehat{}$	effective laminate quantity, e.g. averaged (smeared) stress$\widehat{\sigma}$ or an averaged
$\tilde{}$	denotation of a bending stiffness in the case of a non-symmetrical laminate
$_{comp}$	component
top, bot	top, bottom of a laminate
upp, low	upper, lower surface of a lamina (ply)
^{T}, ^{Tr}, $^{-1}$	transposed, trace, inverse

3 Analyses

3.1 Stress-strain relationship of the uni-directional 3D lamina

Considered is a UD material cube according to Fig. 1 with stresses in the UD lamina coordinate system.

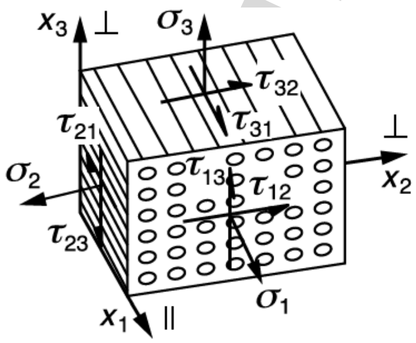


Figure 1: 3D UD lamina stresses with in-plane (intralaminar) stresses and interlaminar stresses (transverse to the lamina plane x_1, x_2)

The following assumptions are made for the UD lamina material:

- The UD lamina is macroscopically homogeneous. It can be treated as a homogenized ('smeared') material
- The stress-strain relationship is linear
- The UD lamina is transversely-isotropic. On planes, parallel with the fiber direction $x_1 \equiv x_{||}$, it behaves orthotropically and on planes transverse to the fiber direction it behaves isotropically.

Under these assumptions and in the case of pure mechanical loading the inverted 3D elasticity matrix $[C]^{-1} = [S]$ (column-normalized, $i = 1, 2, 6)$ with $S_{ij} = S_{ji}$ reads

$$\{\varepsilon\} = [S] \cdot \{\sigma\} = \begin{bmatrix} S_{11} & S_{21} & S_{31} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{32} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \{\sigma\} \ ,$$

and in engineering notation the stress-strain relation

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{21}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{23}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{-\nu_{32}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \qquad (3\text{-}1)$$

where five independent engineering elasticity 'constants' are to be determined. Here, ν_{12} is the larger Poisson's ratio. As the stiffness matrix looks more difficult (see Chapter 6.3) and as the compliance matrix is simpler to use in test data evaluation the compliance matrix above is given for the further use and in the next chapter for 2D stress states, too.
The remaining two constants are mutually dependent due to the fact that in the quasi-isotropic plane $G = E/[(2 \cdot (1+\nu)]$ holds and that further the Maxwell-Betti (reciprocity) relationships can be employed

$$G_{23} = \frac{E_2}{2 \cdot (1+\nu_{23})} = G_{\perp\perp}\ , \quad \nu_{21} = \frac{E_2}{E_1} \cdot \nu_{12}\ , \quad \frac{\nu_{32}}{E_2} = \frac{\nu_{23}}{E_3}. \qquad (3\text{-}2)$$

Using the elasticity matrix of the UD-material is not of any benefit . Nevertheless, the complicated matrix elements were computed and given there for reasons of completeness.

Notes:

1. Check always, by applying the Maxwell-Betti formula, that "The smaller ν times the larger Young's modulus $E_1 = E_{\|}$ is equal to the product of the larger ν times the smaller Young's modulus $E_2 = E_3 = E_{\perp}$". Maxwell-Betti works in the non-linear case, too (see Ref. [18]). One should always take the larger Poisson's ratio in [S] or [C]because the larger one is the better measurable one.

2. Using symbolic suffixes reduces the danger to use a wrong property as input

3. In common literature different sets of indices are used. Hence, in order to avoid another wrong input of a FEM code's material card the actual utilization of the suffixes is to be considered. See VDI 2014, sheet 3, (Ref. [3]) and further in Chapter 5 see the *Note on indexing of Poisson's ratios*.

3.2 In-plane stress-strain relationships of the UD lamina

The relationships above are necessary for a 3D stress-strain analysis. Very often just the in-plane (2D) relationships are needed and this if the CLT is applied. The 3D relationship simplifies for the

in-plane stress state.
After setting the 3 interlaminar stresses zero, stress vector and strain vector read ($1 = \|$, $2 = \perp$)

$$\{\sigma\} = (\sigma_1, \sigma_2, \tau_{12})^T\,, \qquad \{\varepsilon\} = (\varepsilon_1, \varepsilon_2, \gamma_{12})^T. \tag{3-3}$$

Then the strain-stress as well as the stress-strain relationships for the k^{th} lamina reduce from 3D to

$$\{\varepsilon\}_k = [S]_k \cdot \{\sigma\}_k, \quad [S]_k = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{12}}{E_1} & 0 \\ \frac{-\nu_{21}}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}. \tag{3-4}$$

where $[S]_k$ is denoted as reduced (3D→2D) compliance matrix. The first matrix in Eq.3-4 shows the row-normalized indexing (preferred) and the second the column-normalized indexing. In the HSB the larger Poisson's ratio isν_{12}, see also Note in Section 5.

The 2D or in-plane elasticity matrix of the lamina is obtained by setting $\sigma_3 = 0$ in the elasticity matrix $[C]$ (Chapter 6-3) and introducing the relationship into the in-plane stress-strain equations

$$\{\sigma\}_k = [Q]_k \cdot \{\varepsilon\}_k, \quad [Q]_k = \begin{bmatrix} \frac{E_1}{1-\nu_{21}\cdot\nu_{12}} & \frac{\nu_{12}\cdot E_2}{1-\nu_{21}\cdot\nu_{12}} & 0 \\ \frac{\nu_{21}\cdot E_1}{1-\nu_{21}\cdot\nu_{12}} & \frac{E_2}{1-\nu_{21}\cdot\nu_{12}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} = [S]_k^{-1} \tag{3-5}$$

with $[Q]_k$ denoted as reduced stiffness matrix. Fully parallelly, $[S]_k$may be also in a direct way symbolically inverted to obtain $[Q]_k$. Each matrix contains four elasticity constants which are independent from each other.
Above engineering constants or UD material properties should be determined by tests. If micro-mechanical properties for the constituents are available to assess the engineering constants, then the associated micro-mechanical formulas used <u>must</u> be given, too (see respective HSB data sheets).

3.3 Rotation of lamina relationships into the laminate coordinate system

The main axes of the laminate do not generally coincide with those of the lamina, see Fig. 3. For this reason transformation relationships are required for the two constitutive laws. Stress vector and strain vector in the laminate coordinate system read similar as in the lamina coordinate system (see similarity to Eq. (3-5),

$$\{\sigma'\} = (\sigma_x, \sigma_y, \tau_{xy})^T\,, \qquad \{\varepsilon'\} = (\varepsilon_x, \varepsilon_y, \gamma_{xy})^T\,. \tag{3-6}$$

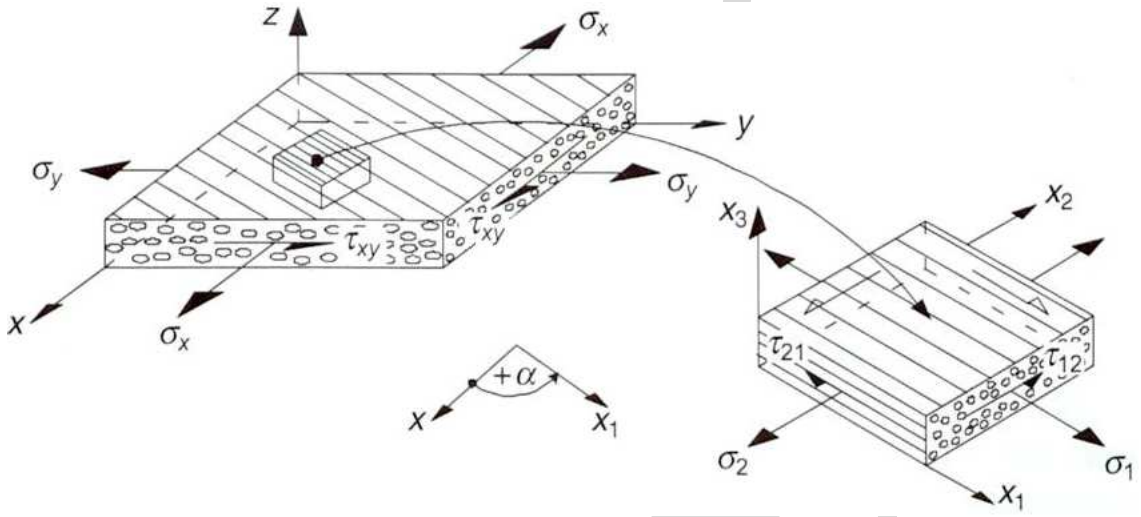


Figure 2: In-plane UD lamina stresses in the laminate COS (left) and in the lamina COS, [VDI 2014]. Definition of the positive fiber orientation angle of the embedded lamina: right-hand system, positively measured from x to fibre direction x_1(not standardized definition)

The stress-strain relationships are yielded as

$$\{\sigma'\} = [T_\sigma] \cdot \{\sigma\} = [T_\sigma] \cdot [Q] \cdot \{\varepsilon\} = [T_\sigma] \cdot [Q] \cdot [T_\varepsilon]^{-1} \{\varepsilon'\} = [Q'] \cdot \{\varepsilon'\}, \tag{3-7}$$

$$\{\varepsilon'\} = [T_\varepsilon] \cdot \{\varepsilon\} = [T_\varepsilon] \cdot [S] \cdot \{\sigma\} = [T_\varepsilon] \cdot [S] \cdot [T_\sigma]^{-1} \{\sigma'\} = [S'] \cdot \{\sigma'\} \tag{3-8}$$

with

$$[T_\sigma] = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -2\cdot\sin\alpha\cdot\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & 2\cdot\sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha & -\sin\alpha\cdot\cos\alpha & (\cos^2\alpha - \sin^2\alpha) \end{bmatrix} \tag{3-9}$$

$$[T_\varepsilon] = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -\sin\alpha\cdot\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ 2\cdot\sin\alpha\cdot\cos\alpha & -2\cdot\sin\alpha\cdot\cos\alpha & (\cos^2\alpha - \sin^2\alpha) \end{bmatrix} \tag{3-10}$$

and with the relationships $[T_\varepsilon]^{-1} = [T_\sigma]^T$, $[T_\sigma]^{-1} = [T_\varepsilon]^T$, by rotation from the laminate COS *x* to the lamina (ply) COS $x_1 = x_{||}$, with α positively measured from $x \rightarrow x_1$.This rotation for material properties means: *What did you off-axis lamina contribute to me, the laminate?* (e.g. Tsai does it oppositely). A transformation $x_1 \rightarrow x$ would mean a rotation by the angle$-\alpha$, a negative angle. Hence after rotation, stiffness and compliance matrices of each single lamina read

$$[Q'] = [T_\sigma] \cdot [Q] \cdot [T_\sigma]^T, \quad [S'] = [T_\varepsilon] \cdot [S] \cdot [T_\varepsilon]^T. \tag{3-11}$$

3.4 The 2D laminate

3.4.1 The constitutive equations

The laminate considered in the following is a plane plate (also applicable to a curved shallow shell) with just an in-plane (intralaminar) stress state. It consists of n UD laminae each having a constant thickness t_k. The lamina is used as computational building block of the laminate. It may describe on one hand several physical layers (plies) or - on the other hand - a part of a layer, see Fig. 3.

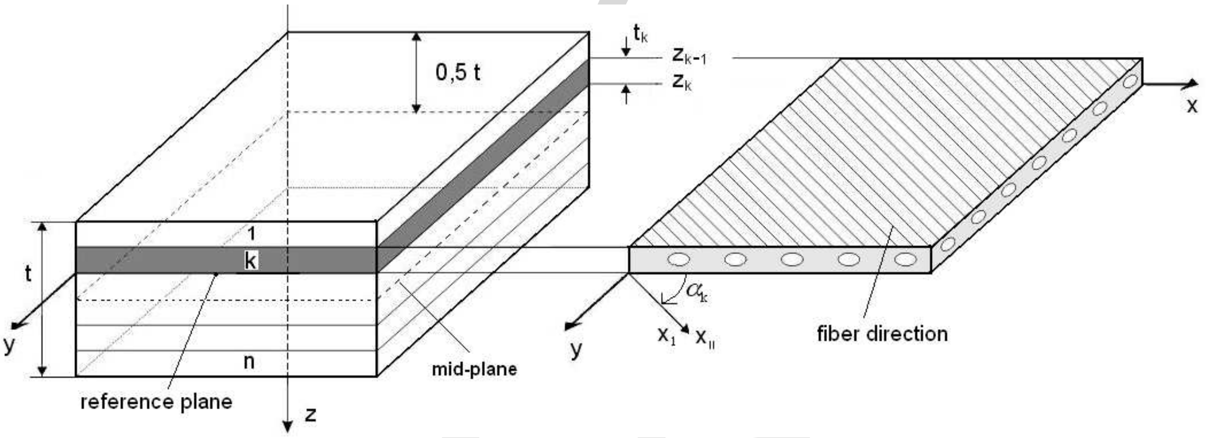


Figure 3: The UD lamina, building block of the laminate.
Here, definition of the positive orientation angle ($x \rightarrow x_1$, clockwise) of the embedded lamina and of a reference plane. In the CLT literature, *z* is usually downward oriented, see also HSB 01200-02 [10]

Each single lamina quantity is rotated by the angle α_k into the lamina COS. The laminate COS is used as reference system.

According to the thin plate theory the following prerequisites must be observed:

- The plate thickness t is small with respect to other dimensions and the transverse deflection of the midplane is small compared to t
- Cross-sections remain planar (Bernoulli hypothesis). This means that the plate is assumed to be rigid to shear stresses
- Displacements perpendicular to the reference plane do not depend on z.

Hence, the strain vector at any point of the plate wall may be written

$$\{\varepsilon'\} = \{\varepsilon^0\} + z \cdot \{\kappa\} \tag{3-12}$$

where $\{\varepsilon^0\}$ and $\{\kappa\}$ represent the complete strain vector of the chosen reference plane. The superscript ^{0} denotes the reference plane, $z = 0$. A distance from the reference plane is denoted z. For hygro-thermal issues apply Ref. [6].

In linear analysis, strains and curvatures of the reference plane, expressed by the displacements $(u^0,\, v^0,\, w^0)$ and the slopes (if linear, small slopes; later used in Annex 6.4) of the reference plane are

$$\{\varepsilon^0\} = \left\{ \begin{array}{c} \dfrac{\partial u^0}{\partial x} \\ \dfrac{\partial v^0}{\partial y} \\ \dfrac{\partial u^0}{\partial y} + \dfrac{\partial v^0}{\partial x} \end{array} \right\}, \qquad \{\kappa\} = \left\{ \begin{array}{c} -\dfrac{\partial^2 w^0}{\partial x^2} \\ -\dfrac{\partial^2 w^0}{\partial y^2} \\ -2\dfrac{\partial^2 w^0}{\partial x \cdot \partial y} \end{array} \right\} = \left\{ \begin{array}{c} -\dfrac{\partial \varphi_x}{\partial x} \\ -\dfrac{\partial \varphi_y}{\partial y^2} \\ -\dfrac{\partial \varphi_y}{\partial x} - \dfrac{\partial \varphi_x}{\partial y} \end{array} \right\}. \tag{3-13}$$

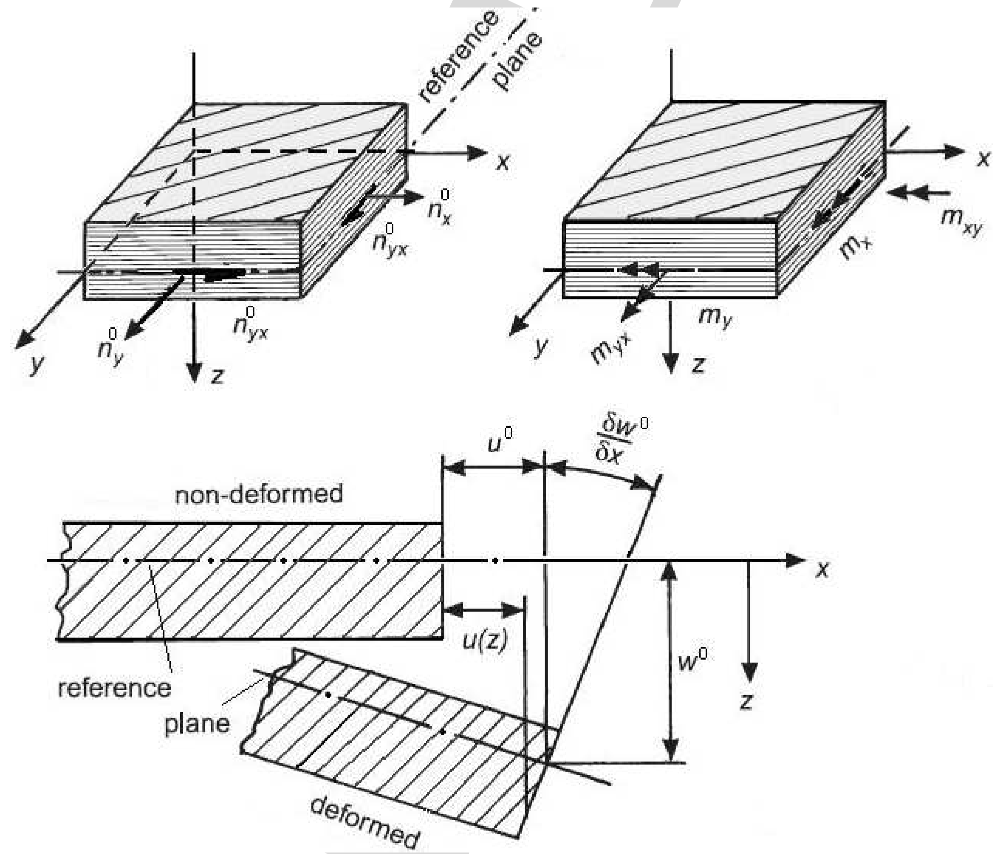


Figure 4: Section forces, section moments and displacements of the laminate plate element.

The general equations of section forces (tractions) and section moments read

$$\{n^0\} = (n_x^0,\, n_y^0,\, n_{xy}^0)^T = \int_{z_{bottom}}^{z_{top}} \{\sigma'\} \cdot dz = \sum_{k=1}^{n} \int_{z_k - 1}^{z_k} \{\sigma'\} \cdot dz\,, \tag{3-14}$$

$$\{m\} = (m_x,\, m_y,\, m_{xy})^T = \int_{z_{bootom}}^{z_{top}} \{\sigma'\} \cdot z \cdot dz\,. \tag{3-15}$$

Utilized units are $n\rightarrow$ N/mm = MPa · mm, $m\rightarrow$N · mm/mm = MPa · mm².

In the case, that all forces are acting in the reference plane and for constant plate thickness, Eq. (3.15) reads

$$\{m\} = (m_x,\, m_y,\, m_{xy})^T = \int_{-t/2}^{t/2} \{\sigma'\} \cdot z \cdot dz = \sum_{k=1}^{n} \int_{z_k-1}^{z_k} \{\sigma'\} \cdot z \cdot dz\,.$$

Values for each of the three section forces and section moments in $\{n^0\}$ and $\{m\}$ are derived from component analysis.

A linear law of elasticity is assumed which means that the following relations are obtained for the laminate, reading in general form,

$$\left\{ \begin{array}{c} n^0 \\ m \end{array} \right\} = \left[\begin{array}{cc} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{array} \right] \cdot \left\{ \begin{array}{c} \varepsilon^0 \\ \kappa \end{array} \right\} = [K] \cdot \left\{ \begin{array}{c} \varepsilon^0 \\ \kappa \end{array} \right\}, \tag{3-16}$$

where $[K]$ is the symmetrical stiffness matrix of the laminate and where i, j = 1, 2, 6 (the 6 considers that τ_{12} is placed at the sixth position of the contracted 3D stress-strain relation, Eq. (3-1)).
The symmetrical sub-matrices ($A_{ij} = A_{ji}$ etc., column-normalized)

$$\boldsymbol{A} = [A] = \left[\begin{array}{ccc} A_{11} & A_{21} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{662} & A_{66} \end{array} \right], \quad [B] = \left[\begin{array}{ccc} B_{11} & B_{21} & B_{61} \\ B_{12} & B_{22} & B_{62} \\ B_{16} & B_{26} & B_{66} \end{array} \right], \quad [D] = \left[\begin{array}{ccc} D_{11} & D_{22} & D_{61} \\ D_{12} & D_{22} & D_{62} \\ D_{12} & D_{26} & D_{66} \end{array} \right]$$

are yielded by summing up the lamina portions

$$[A] = \sum_{k=1}^{n} [Q']_k \cdot t_k = \sum_{k=1}^{n} [Q']_k \cdot (z_k - z_{k-1})\,,$$

$$[B] = \sum_{k=1}^{n} [Q']_k \cdot t_k \cdot \frac{z_k + z_{k-1}}{2} = \sum_{k=1}^{n} [Q']_k \cdot \frac{z_k^2 - z_{k-1}^2}{2}, \tag{3-17}$$

$$[D] = \sum_{k=1}^{n} [Q']_k \cdot \left[t_k^3/12 + t_k \cdot \left(\frac{z_k + z_{k-1}}{2} \right)^2 \right] = \sum_{k=1}^{n} [Q']_k \cdot \frac{z_k^3 - z_{k-1}^3}{3}$$

.

The submatrices are termed extensional (membrane) stiffness matrix [A], coupling stiffness matrix [B] and bending (flexural) stiffness matrix [D].
In the *isotropic* case reads: $A_{11} = A_{22} = E \cdot t/(1-\nu^2),\ D_{11} = D_{22} = E \cdot t^3/(12 \cdot (1-\nu^2))$.
Units used are: MPa · mm for A_{ij}, MPa · mm² for B_{ij}, MPa · mm³ for D_{ij}.
Counting in z-direction, in the Eqs. (3-17) z_k denotes the distance of the more far surface of the k^{th} layer from the reference plane $z = 0$ and z_{k-1} denotes the difference of the surface of the k^{th} layer less far from the reference plane, $z_k > z_{k-1}$, (see Fig. 3).

The stiffness matrix $[K]$ is dependent on the chosen reference plane. In the case of a symmetric stack, the geometrical mid-plane is usually selected as reference plane.
Due to VDI 2014 : In the general anisotropic case all positions of the laminate stiffness matrix are filled in. In the special case of an asymmetrical lay-up with orthotropy $[\alpha/-\alpha]$ or $[\alpha/-\alpha/\alpha/-\alpha]$ the terms $A_{16},\ A_{26},\ B_{11}, B_{12},\ B_{22},\ B_{66},\ D_{16},\ D_{26}$ disappear

$$[K] = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix}, \tag{3-18}$$

inserting $A_{ij} = A_{ji}$. The zero value of the terms A_{16} and A_{26} is a consequence of orthotropy. Quasi-isotropy is given for the stacks [0/90/45/-45], [-60/0/60] and its permutations.
In the special case of symmetrical lay-up with orthotropy – also called the classic or centrically orthotropic case – the shear terms A_{16} and A_{26}, the torsion terms D_{16} and D_{26} and also the entire bending-stretching coupling matrix [*B*] takes the value of zero. Coupling effects exist, firstly in shear-strain coupling when A_{16} and A_{26} are not equal to zero, and secondly in bending-twisting coupling when D_{16} and D_{26} are not equal to zero. In the first case a normal force will then simultaneously cause shear (for example in a UD tensile specimen with $\alpha \neq 0$ or $\neq 90$. And, in the second case a bending moment will cause twisting (for example, a laminate test specimen with a lay-up of $[45/-45]_s$). Annex 6.3 will depict the effect of the *K*-matrix elements on the deformation of the laminate material element.
Whereas the A_{ik} do not depend on the chosen reference plane the B_{ik} and D_{ik} do. Therefore, the size of the B_{ik} and D_{ik} is not equivalent to their influence.
Flexural-torsional coupling can be significant in plates with a general, anisotropic or even with an orthotropic structure.

Notes:
(a) From the point of view to better understand mechanics and to simplify calculation, it would be good to have a common neutral surface of a plate (where $[B] = 0$). As a result of $\{m\}$, the loading-dependent strains $\varepsilon_x,\ \varepsilon_y$ and γ_{xy} are zero, which permits in-plane or laterally loaded plate problems to be considered separately from each other. However, except for symmetric lay-ups a common neutral plane does not exist: For example, laminates that have the following eccentrically orthotropic structure [0/0/90/0/0/90] possess different neutral planes for the (x, z)- and the (y, z) cross-section and in consequence no analytical or numerical simplification can be achieved.
(b) For a simpler understanding the following equations for the isotropic case are given below

$$\begin{Bmatrix} m_x \\ m_y \\ m_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \cdot \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \frac{t^3 \cdot E}{12 \cdot (1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \cdot \{\kappa\}\,. \tag{3-19}$$

Examples for boundary conditions

Of the many combinations of boundary conditions two examples are presented, just principally:

***(1) Inverse task*:** In most of the practical cases, the loading is given and the inverse equation to Eq. (3-16) is required

$$\left\{ \begin{matrix} \varepsilon^0 \\ \kappa \end{matrix} \right\} = [K]^{-1} \cdot \left\{ \begin{matrix} n^0 \\ m \end{matrix} \right\} = \left[\begin{matrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{matrix} \right]^{-1} \cdot \left\{ \begin{matrix} n^0 \\ m \end{matrix} \right\} = \left[\begin{matrix} \boldsymbol{A}^* & \boldsymbol{B}^* \\ (\boldsymbol{B}^*)^T & \boldsymbol{D}^* \end{matrix} \right] \cdot \left\{ \begin{matrix} n^0 \\ m \end{matrix} \right\}, \quad (3\text{-}20)$$

$$= \left[\begin{matrix} (\boldsymbol{A} - \boldsymbol{B}^T \cdot \boldsymbol{D}^{-1} \cdot \boldsymbol{B})^{-1} & (\boldsymbol{B} - \boldsymbol{D} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{A})^{-1} \\ (\boldsymbol{B} - \boldsymbol{D} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{A})^{-1} & (\boldsymbol{D} - \boldsymbol{B} \cdot \boldsymbol{A}^{-1} \cdot \boldsymbol{B})^{-1} \end{matrix} \right] \cdot \left\{ \begin{matrix} n^0 \\ m \end{matrix} \right\}$$

with $[K]^{-1}$ as inverse laminate stiffness matrix, usually termed laminate compliance matrix.
Note: In [5], Tsai differently uses the asterix to mark the normalization of the sub-matrices [A], [B], and [D] in order to obtain the same unit for all stiffnesses in [K].

***(2) Mixed task*:** If load conditions as well as boundary conditions are given, “mixed” relationships must be considered by a semi-inverted form

$$\left\{ \begin{matrix} \varepsilon^0 \\ m \end{matrix} \right\} = \left[\begin{matrix} \breve{\boldsymbol{A}} & \breve{\boldsymbol{B}} \\ (-\breve{\boldsymbol{B}})^T & \breve{\boldsymbol{D}} \end{matrix} \right] \cdot \left\{ \begin{matrix} n^0 \\ \kappa \end{matrix} \right\} = \left[\begin{matrix} \boldsymbol{A}^{-1} & -\boldsymbol{B} \cdot \boldsymbol{A}^{-1} \\ \boldsymbol{B} \cdot \boldsymbol{A}^{-1} & \boldsymbol{D} - \boldsymbol{B} \cdot \boldsymbol{A}^{-1} \cdot \boldsymbol{B} \end{matrix} \right] \cdot \left\{ \begin{matrix} n^0 \\ \kappa \end{matrix} \right\}. \quad (3\text{-}21)$$

Generally, the matrices $\breve{\boldsymbol{A}}$ and $\breve{\boldsymbol{D}}$ are symmetric while $\breve{\boldsymbol{B}}$ is not.

Strains and stresses in lamina COS

For the strength analysis the so-called natural or lamina stresses and strains in the lamina COS are derived from above stresses and strains by employing the Eqs. (3-5, **??** and 3-12):

$$\{\sigma\}_k = [Q]_k \cdot \{\varepsilon\}_k, \quad \{\varepsilon\}_k = [T_\varepsilon]_k^{-1} \cdot \{\varepsilon'\}_k = [T_\sigma]_k^T \cdot \{\{\varepsilon^0\} + z_k \cdot \{\kappa\}\}\ . \quad (3\text{-}22)$$

In order to fully check those laminae which turned out to might be critical, Eq.(3.22) is applied to both the lamina surfaces, to $z_{k-1}$and z_k, in order to find the critical stresses or strains in the laminate. Often, the worse side is still known from mechanical considerations.

3.4.2 Effective elasticity ’constants’ and effective stiffnesses of laminates

Sometimes the effort with the laminate analysis can be reduced by applying a simpler, homogenized model which is elasticity-equal. For instance, if a direct input is necessary for the use of an analytical formula or for a finite-element laminate shell analysis, then elasticity ’constants’ are required or equivalent laminate stiffnesses (see further the HSBs 32520 on beams). This may be the case for a plate (unit width “1”) or for a strip (b≫t). These properties shall be computed now by using the CLT plate theory. In order to show the transition via the strip to a laminated beam the corresponding beam formulas are also given in the Annex.

Effective elasticity constants and effective stiffnesses are derived by comparing the stiffnesses of the ’homogenized’ laminate cross-section with the non-homogeneous cross-section via the relations for the laminate’s section forces and section moments (ϑ is the specific angle of twist)

$n = \widehat{Et} \cdot \varepsilon, \quad m = (EI)_{ef} \cdot \kappa = \widehat{EI} \cdot \kappa$ and the torsion (twisting) moment

$$m_{xy} = D_{66} \cdot \kappa_{xy} \quad \text{to be related to} \quad m_t = \widehat{GI_t} \cdot \vartheta'.$$

The hat marks that the derived stiffness property is a 'smeared' laminate quantity. The provision of effective elasticity (engineering) constants makes only sense for $[B] = 0$. Otherwise irritation will be faced by the user, see Ref. [8]. Basis for the derivation of above quantities is the general laminate stiffness matrix $[K]$, Eq. (3-18).

(a) Effective elasticity constants ('engineering constants') of the membrane, $[B] = 0$

Elasticity constants (see [8]) are effective (index$_{ef}$) values if the cross-section is not homogeneous. They are also termed 'engineering constants' but they are constants, only, as far as their initial values are addressed. The provision of engineering constants makes sense for orthotropic symmetric laminates, only, where $B_{ik} = 0$ and $A_{16} = A_{26} = 0$.

<u>Procedure via stiffnesses:</u>
The used section forces-strain relationship reads

$$\begin{Bmatrix} n_x^0 \\ n_y^0 \\ n_{xy}^0 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}. \tag{3-23}$$

For the plate, subjected to $n_y^0 = n_{xy}^0 = 0, n_x^0 = \hat{E}_x \cdot t \cdot \varepsilon_x^0$ ("membrane"), the following two effective elasticity constants of the laminate are firstly derived from the equations above

$$\begin{aligned} \widehat{E_x}: &\quad n_x^0 = A_{11} \cdot \varepsilon_x^0 + A_{12} \cdot \varepsilon_y^0 \quad \text{and} \quad 0 = A_{12} \cdot \varepsilon_x^0 + A_{22} \cdot \varepsilon_y^0 \\ &\quad n_x^0 = \hat{E}_x \cdot t \cdot \varepsilon_x^0 \equiv \varepsilon_x^0 \cdot (A_{11} - A_{12}^2 / A_{22}) \qquad , \\ \widehat{\nu_{xy}}: &\quad 0 = A_{12} \cdot \varepsilon_x^0 + A_{22} \cdot \varepsilon_y^0 \quad \Rightarrow \widehat{\nu_{xy}} = -\varepsilon_y^0 / \varepsilon_x^0 = A_{12} / A_{22} \end{aligned} \tag{3-24}$$

and secondly for the plate, subjected to $n_x^0 = n_y^0 = 0, n_{xy}^0 = \widehat{G_{xy}} \cdot t \cdot "1" \cdot \gamma_{xy}^0$

$$\widehat{G_{xy}}: \quad n_{xy}^0 = \widehat{G_{xy}} \cdot t \cdot \gamma_{xy}^0 \quad \Rightarrow \widehat{G_{xy}} \cdot t \cdot \gamma_{xy}^0 = \gamma_{xy}^0 \cdot A_{66}.$$

Eventually, all effective elasticity constants read:

$$\widehat{E_x} := \frac{A_{11} - A_{12}^2/A_{22}}{t} = \frac{1}{t \cdot A_{11}^*}, \ \widehat{E_y} = \frac{A_{22} - A_{12}^2/A_{11}}{t}, \ \widehat{\mathrm{G_{xy}}} = \frac{\mathrm{A_{66}}}{\mathrm{t}}, \widehat{\nu_{xy}} = +\frac{A_{12}}{A_{22}}. \tag{3-25}$$

In HSB 37155-01, for some UD lamina-composed laminates the dependencies of the in-plane elasticity constants are shown.

For the *isotropic plate* the elasticity constants simplify to ($A_{ik} = Q_{ik} \cdot t$)

$$\widehat{E_x} = E_x = \frac{A_{11} - A_{12}^2/A_{22}}{t} = \frac{Q_{11} - Q_{12}^2/Q_{22}}{t} \cdot t = \frac{E}{1-\nu^2} - \frac{E \cdot \nu^2}{1-\nu^2} = E \cdot \frac{1-\nu^2}{1-\nu^2} = E.$$

<u>Procedure via compliances:</u>
This formalistically and numerically simpler procedure uses the stiffness matrix that is necessarily

to be inverted before. Generally, inserting sub-matrix symmetry relations of indices, both the matrices read

$$[K] = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \Rightarrow \begin{bmatrix} A^*_{11} & A^*_{12} & A^*_{16} & B^*_{11} & B^*_{12} & B^*_{16} \\ A^*_{12} & A^*_{22} & A^*_{26} & B^*_{12} & B^*_{22} & B^*_{26} \\ A^*_{16} & A^*_{26} & A^*_{66} & B^*_{16} & B^*_{26} & B^*_{66} \\ B^{*T}_{11} & B^{*T}_{12} & B^{*T}_{16} & D^*_{11} & D^*_{12} & D^*_{16} \\ B^{*T}_{12} & B^{*T}_{22} & B^{*T}_{26} & D^*_{12} & D^*_{22} & D^*_{26} \\ B^{*T}_{16} & B^{*T}_{26} & B^{*T}_{66} & D^*_{16} & D^*_{26} & D^*_{66} \end{bmatrix} = [K]^{-1}, \tag{3-26}$$

which leads for $[B] = 0]$ and $A_{16} = A_{26} = 0$ to the relations

$$\widehat{E_x} := \frac{1}{A^*_{11} \cdot t} = \frac{A_{11} - A^2_{12}/A_{22}}{t}, \ \widehat{E_y} = \frac{1}{A^*_{22} \cdot t}, \ \widehat{\mathrm{G_{xy}}} = \frac{1}{\mathrm{A^*_{66} \cdot t}}, \ \widehat{\nu_{xy}} = -\frac{A^*_{12}}{A^*_{11}} = \frac{A_{12}}{A_{22}}. \tag{3-27}$$

(b) Effective bending and torsion stiffnesses (rigidities) of the symmetric laminate

The use of specific effective elastic constants for bending and twisting makes no sense, in general, however the use of effective stiffnesses. These are determined for *symmetric laminates*, only, where $[B] = 0$, $D_{16} = D_{26} = 0$. Then the bending equations reduce to

$$\begin{Bmatrix} m_x \\ m_y \\ m_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \cdot \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \tag{3-28}$$

The size of the D_{ik}elements is linked to the chosen reference plane.

(I) Curvature-constrained wide plate, using stiffnesses

Considering, that due to the large width the plate is curvature-constrained in y-direction ($\kappa_y = 0$) and that it is subjected to $m = EI_{ef} \cdot \kappa = \widehat{EI}_x \cdot \kappa$ exemplarily is derived

$$\widehat{EI_x}: \qquad m_x = D_{11} \cdot \kappa_x + D_{12} \cdot 0 = \widehat{EI_x} \cdot \kappa_x \quad .$$

The torsional moment (torque) required for unit twist is termed (effective) torsional stiffness or rigidity $\widehat{GI_t}$. How the twisting size responsible torsional stiffness (rigidity) $\widehat{GI_t}$ is obtained, this is derived in Annex 6.4. In contrast to two themselves balancing two bending moments four balancing twisting moments must be considered for a plate. They collectively build up the *internal twisting* (for details, see Annex 6.4). All effective laminate stiffnesses then read

$$\widehat{EI_x} = D_{11}, \ \widehat{EI_y} = D_{22}, \quad \widehat{GI_t} = 4 \cdot D_{66}. \tag{3-29}$$

Note: The way via the compliances , e.g. $1/D^*_{11} = (D_{11} - D^2_{12}/D_{22})$, does mathematically not allow to use ($\kappa_y = 0$) because this would lead to $m_y \neq 0$.

For the *isotropic plate* the stiffness elements simplify to

$$D_{ik} = Q_{ik} \cdot \frac{(0.5 \cdot t)^3 - (-0.5 \cdot t)^3}{3} = Q_{ik} \cdot \frac{t^3}{12}, \; D_{11} = \frac{E}{1-\nu^2} \cdot \frac{t^3}{12}, \; D_{22} = D_{11}, \quad D_{12} = \nu \cdot D_{11}$$

$$D_{66} = 0.5 \cdot D_{11} \cdot (1-\nu) = 0.5 \cdot \frac{E}{1-\nu^2} \cdot \frac{t^3}{12} \cdot (1-\nu) = G \cdot t^3/12 \quad , \quad ,$$

$$\widehat{EI_x} = D_{11} = Q_{11} \cdot \frac{t^3}{12} \cdot \left(1-\nu^2\right) = E \cdot t^3/[12 \cdot \left(1-\nu^2\right)]; \widehat{GI_t} = 4 \cdot D_{66} = G \cdot t^3/3.$$

(II) Free wide plate, using compliances

In the general non-symmetric case the application of the curvature constraints ($\kappa_y = 0$) to determine $\widehat{EI_x}$ and of ($\kappa_x = 0$) to determine $\widehat{EI_y}$ would lead to an over-determination. Curvature constraints cannot be regarded anymore. Therefore, the inverse $1/D^*_{ii}$ is used accepting lower stiffness values

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} A^*_{11} & A^*_{12} & A^*_{16} & B^*_{11} & B^*_{12} & B^*_{16} \\ A^*_{12} & A^*_{22} & A^*_{26} & B^*_{12} & B^*_{22} & B^*_{26} \\ A^*_{16} & A^*_{26} & A^*_{66} & B^*_{16} & B^*_{26} & B^*_{66} \\ B^{*T}_{11} & B^{*T}_{12} & B^{*T}_{16} & D^*_{11} & D^*_{12} & D^*_{16} \\ B^{*T}_{12} & B^{*T}_{22} & B^{*T}_{26} & D^*_{12} & D^*_{22} & D^*_{26} \\ B^{*T}_{16} & B^{*T}_{26} & B^{*T}_{66} & D^*_{16} & D^*_{26} & D^*_{66} \end{bmatrix} \cdot \begin{Bmatrix} n_x^0 \\ n_y^0 \\ n_{xy}^0 \\ m_x \\ m_y \\ m_{xy} \end{Bmatrix} \quad (3\text{-}30)$$

.

<u>Notes</u> :

(1) The analytical way *(I)* above, using stiffnesses, is intentionally gone with respect that the HSB-sheets primarily use stiffnesses. Way *(II)* is applying compliances (= elements of the inverted stiffness matrix K^{-1}) and thereby the use of simpler relationships, for instance $\widehat{EI_x} = 1/D^*_{11}$. However, $1/D^*_{ii} < D_{ii}$ is on the safe side.

(2) $[B] \neq 0$: In this case, membrane loading cannot be separated from bending. Therefore, the derivation of effective elasticity constants principally makes no sense. Exemplarily, a 3-lamina-composed symmetric laminate [45/-45$_2$/45]

(3) Poisson's ratio ν: In comparison to metals the larger Poisson ratio may be larger than 0.3 (for [45/-45/-45/45] $\nu_{xy} = \nu_{yx} = 0.8$) and also smaller than 0.3. Of course, this is moduli-dependent. Further, the product of the associated Poisson ratios $(\nu_{xy} \cdot \nu_{yx})$ may be smaller than 0.3^2. Further see Annex 6.3.

(c) Stiffnesses of the non-symmetrical laminate $\{n^0\} = 0$

For the often utilized balanced laminates extension-shear coupling is zero ($A_{16} = A_{26} = 0$), if for instance 45°-layers are used such as in the non-symmetric stack [0/90/45/-45/] of the Example 4.8, however the bending stiffnesses do not vanish, i.e. $D_{16} \neq 0$ and $D_{26} \neq 0$, This means, that bending-torsion coupling occurs and care must be taken to this effect due to Eq. (3-18).

In special cases a global information on the laminate stiffnesses may be desired (despite of $[B] \neq 0$) for an analytical approximate calculation. Then, 'effective' stiffnesses D_{ik} of a non-symmetrical laminate are calculated according to Eq. (3-30) and the obtained bending stiffnesses are comparison "numbers", only.

If $\{n\} = 0$ and the coupling stiffnesses do not vanish, then the membrane strains $\{\varepsilon^0\}$ can be replaced by the curvatures $\{\kappa\}$. In consequence, the bending moments are just linked to the curvatures, whereby the associated stiffnesses become the so-called modified bending stiffnesses $[\tilde{D}]$ ($\equiv [\breve{D}]$)

$$[\tilde{D}] = [D] - [B] \cdot [A]^{-1} \cdot [B] \,. \tag{3-31}$$

The $\tilde{D}-$stiffnesses are independent of the choice of the reference plane. For symmetric orthotropic laminates $[\tilde{D}] = [D]$ is valid, due to $B_{ij} = 0$. For further details, see Example 4.8.

Note: Bending and torsion of plates can be well described by $[\tilde{D}]$. Is there a need to tackle the stability problem by a geometrically non-linear analysis then the full K-matrix should be used.

3.4.3 Loadings and coordinate system

Of special interest is the reference plane and the lay-up definition. In principle, it can be chosen as suitable as possible. On the other side, also the loading may be provided arbitrarily at different planes, view Fig. 5.
Due to that fact, the analyst needs to harmonize the input.
In Annex 6.4 much effort has been put on the effect of the action plane of the section forces.

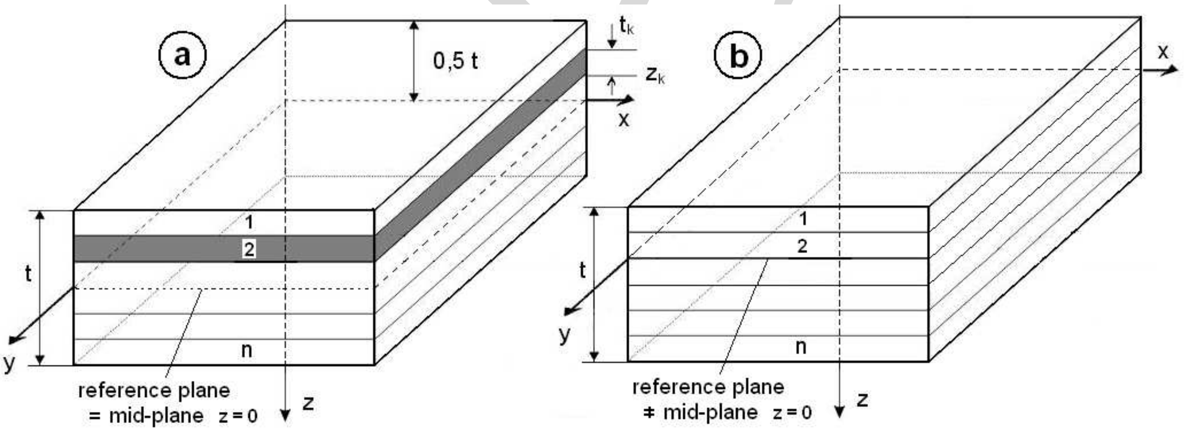


Figure 5: Examples for chosen reference planes,

4 Examples

In order to illustrate how the stack affects laminate stiffness matrix K and lamina stresses σ two sets of use cases are presented. The first set refers to plates and the second to a tube. According to their rotational symmetry a tube cannot experience curvatures κ. This simplifies the computational work. Residual stresses and strength analysis are not addressed. As different units (percent and micro-strain) are used for the strains this is applied to in the tables of results.
Note: For a simpler comparison of data the numbers for the stresses, strains, and elasticity constants are rounded up when citing. The rounded final result values are based on the original values of higher numerical precision.

Given Materials:

Two typical FRP/Epoxy UD material data sets, taken from HSB 37106-01 (moisture, [13]) and HSB 37110-01, [9].

* Lamina data: $V_f = 60\%$, the units for the α_M are written in a manner, similar to α_T. The 2D analysis requires 4 UD elasticity properties

* For the single layer ($\equiv$ physical lamina) of the stack the following properties are given with the symbolic subscripts being related as $_1 \equiv {}_{\|}$, $_2 \equiv {}_{\perp}$ (experience proved that the use of symbolic indexes helps to avoid input mistakes with the properties):

	$E_\|$	$E_\perp$	$G_{\|\perp}$	$\nu_{\|\perp}$	$\alpha_{T\|}$	$\alpha_{T\perp}$	$\alpha_{M\|}$	$\alpha_{M\perp}$
	MPa	MPa	MPa	–	$10^{-6}\cdot$	mm/(mm·K)	$10^{-4}\cdot$	mm/(mm·%)
GFRP	45200	11600	4500	0.27	6	22	0	55
CFRP	132700	9300	4600	0.28	0.23	29	1	42

Configuration:

z-coordinate and numbering of layers 'downward', x-coordinate corresponds to 0°, see Fig. 3.

4.1 CFRP: Symmetric stack $[0/90/90/0] \equiv [0/90]_s$, $t = 1.0\,\text{mm}$, $t_k = 0.25\,\text{mm}$

Task: Determination of stiffness matrix, lamina strains and stresses (necessary for a later strength analysis). The computation is linear. Two variants of reference planes (a), (b) are investigated.

Computation:

Using Eq. (3-5), with the properties above at first the stiffness (elasticity) matrix of the UD material is computed. It reads in the (local) lamina coordinate system

$$[Q] = \begin{bmatrix} 133433 & 2618 & 0 \\ 2618 & 9351 & 0 \\ 0 & 0 & 4600 \end{bmatrix} \text{MPa}\,.$$

In order to achieve the contribution of each lamina to the laminate stiffness the stiffness matrix of each lamina has to be transformed into the laminate COS by the transformation matrices

$$[T_\sigma]_2 = [T_\sigma]_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad [T_\sigma]_1 = [T_\sigma]_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Eq. (3-11) delivers in the (global) laminate coordinate system

$$[Q']_1 = [Q']_4 = \begin{bmatrix} 133433 & 2618 & 0 \\ 2618 & 9351 & 0 \\ 0 & 0 & 4600 \end{bmatrix} \mathrm{MPa}, [Q']_2 = [Q']_3 = \begin{bmatrix} 9351 & 2618 & 0 \\ 2618 & 133433 & 0 \\ 0 & 0 & 4600 \end{bmatrix} \mathrm{MPa}.$$

<u>Variant (a)</u> : loading acts in mid-plane, midplane is chosen as reference plane

Given: Loading $\{n^0\} = (300,\, 0,\, 0)^T\ \mathrm{N/mm}, \quad \{m\} = (50,\, 0,\, 0)^T\ \mathrm{N{\cdot}mm/mm}$,Fig. 5a

The laminate stiffness matrix, applying Eqs. (3-17, 3-18), reads

$$[K] = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} 71392 & 2618 & 0 & 0 & 0 & 0 \\ 2618 & 71392 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4600 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9827 & 218.2 & 0 \\ 0 & 0 & 0 & 218.2 & 2072 & 0 \\ 0 & 0 & 0 & 0 & 0 & 383.3 \end{bmatrix}$$

with [*A*] in $\mathrm{MPa \cdot mm}$= N/mm, [*B*] in $\mathrm{MPa \cdot mm^2 = N}$, [*D*] in $\mathrm{MPa \cdot mm^3}$ and its inverse

$$[K]^{-1} = \begin{bmatrix} 1.40 & -0.05 & 0 & 0 & 0 & 0 \\ -0.05 & 1.40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 21.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10.2 & -1.07 & 0 \\ 0 & 0 & 0 & -1.07 & 48.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 260.9 \end{bmatrix} \cdot 10^{-5}.$$

Using the Eqs.(3-25), the effective 'elasticity constants' of the laminate (membrane loading) are computed via stiffnesses and via compliances as

$$\widehat{E_x} = \frac{A_{11} - A_{12}^2/A_{22}}{t} = (71392 - 2613^2/71392)/1.00\,\mathrm{MPa} = 71296\,\mathrm{MPa}$$

$$\equiv \widehat{E_x} = \frac{1}{A_{11}^* \cdot t} = 1/(1.40 \cdot 10^{-5} \cdot 1.00)\,\mathrm{MPa} = 71296\,\mathrm{MPa}.$$

$$\widehat{E_x} = 71296\,\mathrm{MPa} = \widehat{E_y}, \quad \widehat{G_{xy}} = 4600\,\mathrm{MPa}, \quad \widehat{\nu_{xy}} = -\frac{-0.05}{1.40} = 0.037 = \widehat{\nu_{yx}}.$$

Effective laminate stiffnesses per width under bending and torsion follow from Eqs. (3-29 or 3-30)

$$\widehat{EI_x} = D_{11} = 9827\,\mathrm{MPa \cdot mm^3} > D_{11}^{*-1}, \widehat{EI_y} = 2072\,\mathrm{MPa \cdot mm^3}, \widehat{GI_t} = 4 \cdot 383.3\,\mathrm{MPa \cdot mm^3}.$$

Due to Eq. (3-22), the strains within the mid-plane are calculated as

$$\{\varepsilon^0\} = \begin{Bmatrix} 4.2 \\ -0.2 \\ 0 \end{Bmatrix} \cdot 10^{-3}\,, \qquad \{\kappa\} = \begin{Bmatrix} 5.1 \\ -0.5 \\ 0 \end{Bmatrix} \cdot 10^{-3} \cdot 1/\mathrm{mm}\,.$$

Herewith, the vector of strains at the boundary surfaces of the laminae can be determined with Eq. (3-12)

layer number	1	1	2	2	3	3	4	4
fiber angle	0°	0°	90°	90°	90°	90°	0°	0°
z in mm	-0.5	-0.25	-0.25	0	0	0.25	0.25	0.5
ε_x in $10^{-3} \cdot$mm/mm	1.7	2.9	2.9	4.2	4.2	5.5	5.5	6.8
ε_y in $10^{-3} \cdot$mm/mm	0.1	0	0	- 0.2	-0.2	-0.3	- 0.3	-0.4
γ_{yx} in $10^{-3} \cdot$mm/mm	0	0	0	0	0	0	0	0

With the application of Eq. (3-21) follow the lamina strains and stresses in the four laminae (in order to use examples as a numerical benchmark the numbers are not rounded)

z in mm	-0.5	-0.25	-0.25	0	0	0.25	0.25	0.5
$\varepsilon_{\|\|}$ in $10^{-3} \cdot$mm/mm	1.7	2.9	0	-0.2	-0.2	-0.3	5.5	6.8
$\varepsilon_\perp$ in $10^{-3} \cdot$mm/mm	0.1	0	2.9	4.2	4.2	5.5	-0.3	-0.4
$\gamma_{\perp\|\|}$ in $10^{-3} \cdot$mm/mm	0	0	0	0	0	0	0	0
$\sigma_{\|\|}$ in MPa	221.5	391.3	5.0	-9.6	-9.6	-24.2	730.8	900.6
$\sigma_\perp$ in MPa	5.4	7.5	27.4	38.9	38.9	50.5	11.7	13.7
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	0	0	0	0

Variant (b) : loading acts in mid-plane, top surface is chosen as reference plane

Given: Loading (for the consideration of the section force offset, see Annex 6.1)

$\{n^0\} = (300,\,0,\,0)^T\,\mathrm{N/mm}$, $\{m\} = (50 + z_{n_x} \cdot 300,\,0,\,0)\,\mathrm{N{\cdot}mm/mm}$, $z_{off\,x} = 0.25\,\mathrm{mm}$.

Loading is analogous to *Fig. 6*b. The midplane force is acting at a distance of $+0.25$mm to the reference plane. This is considered in the reference plane by an the additional moment above.

The laminate stiffness matrix, applying Eq. (3-17), reads

$$[K] = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} 71392 & 2618 & 0 & 17848 & 655 & 0 \\ 2618 & 71392 & 0 & 655 & 17848 & 0 \\ 0 & 0 & 4600 & 0 & 0 & 1150 \\ 17848 & 655 & 0 & 14289 & 382 & 0 \\ 655 & 17848 & 0 & 382 & 6534 & 0 \\ 0 & 0 & 1150 & 0 & 0 & 671 \end{bmatrix},$$

with [A] in $\mathrm{MPa \cdot mm}$, [B] in $\mathrm{MPa \cdot mm^2 = N}$, [$D$] in $\mathrm{MPa \cdot mm^3}$. Its inverse is

$$[K]^{-1} = \begin{bmatrix} 2.04 & -0.12 & 0 & -2.55 & 0.27 & 0 \\ -0.12 & 4.43 & 0 & 0.27 & -12.1 & 0 \\ 0 & 0 & 38.0 & 0 & 0 & -65.2 \\ -2.55 & 0.27 & 0 & 10.2 & -1.07 & 0 \\ 0.27 & -12.1 & 0 & -1.07 & 48.4 & 0 \\ 0 & 0 & -65.2 & 0 & 0 & 260.9 \end{bmatrix} \cdot 10^{-5} .$$

The effective elasticity constants are as in variant (a) but the effective laminate stiffnesses read
$\widehat{EI_x} = 14289\,\mathrm{MPa}\cdot\mathrm{mm}^3$, $\widehat{EI_y} = 6534\,\mathrm{MPa}\cdot\mathrm{mm}^3$, $\widehat{GI_t} = 4\cdot 671\,\mathrm{MPa}\cdot\mathrm{mm}^3$.

According to Eq. (3-18) the strains of the reference plane are calculated

$$\{\varepsilon^0\} = \begin{Bmatrix} 2.9 \\ 0 \\ 0 \end{Bmatrix} \cdot 10^{-3} , \qquad \{\kappa\} = \begin{Bmatrix} 5.1 \\ -0.5 \\ 0 \end{Bmatrix} \cdot 10^{-3} \cdot 1/\mathrm{mm} .$$

Herewith, the vector of strains $\{\varepsilon'\}$ at the boundary surfaces of the laminae can be determined with Eq. (3-12)

layer number	1	1	2	2	3	3	4	4
fiber angle	0°	0°	90°	90°	90°	90°	0°	0°
z in mm	−0.25	0	0	0.25	0.25	0.5	0.5	0.75
ε_x in $10^{-3}\cdot$mm/mm	1.7	2.9	0	-0.2	−0.2	−0.3	5.5	6.8
ε_y in $10^{-3}\cdot$mm/mm	0.1	0	2.9	4.2	4.2	5.5	- 0.3	−0.4
γ_{yx} in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0

With the application of Eq. (3-22) follow the lamina strains and stresses in the four laminae

z in mm	−0.25	0	0	0.25	0.25	0.5	0.5	0.75
$\varepsilon_{\|\|}$ in $10^{-3}\cdot$mm/mm	1.7	2.9	0	−0.2	−0.2	−0.3	5.5	6.8
$\varepsilon_{\perp}$ in $10^{-3}\cdot$mm/mm	0.1	0	2.9	4.2	4.2	5.5	-0.3	−0.4
$\gamma_{\perp\|\|}$ in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0
$\sigma_{\|\|}$ in MPa	221.5	391.3	5.0	-9.6	-9.6	-24.2	730.8	900.6
$\sigma_{\perp}$ in MPa	5.4	7.5	27.4	38.9	38.9	50.5	11.7	13.7
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	0	0	0	0

<u>Notes on the effect of differently chosen reference planes in Variant (a) and (b):</u>

The stress and strain results do not depend on the choice of the reference plane. For clarification, exemplarily for κ_x, a proof shall be analytically given:

(Variant a)
$\kappa_x = m_x/D_{11} = (50\,\mathrm{N}\cdot\mathrm{mm/mm})/9827\,\mathrm{N}\cdot\mathrm{mm} = 5.1\cdot 10^{-3}\,\mathrm{mm}^{-1} \equiv 50\cdot 10.2\cdot 10^{-5}\,\mathrm{mm}^{-1}$

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} 1.40 & -0.05 & 0 & 0 & 0 & 0 \\ -0.05 & 1.40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 21.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10.2 & -1.07 & 0 \\ 0 & 0 & 0 & -1.07 & 48.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 260.9 \end{bmatrix} \cdot 10^{-5} \cdot \begin{Bmatrix} 300 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \end{Bmatrix} ,$$

(Variant b)
$\kappa_x = B_{11} \cdot n_x^0 + D_{11} \cdot (m_x + z_{offx} \cdot n_x^0) =$
$(-2.55 \cdot 10^{-5}/\mathrm{N}) \cdot 300\,\mathrm{N/mm} + (10.2 \cdot 10^{-5}/\mathrm{N\cdot mm}) \cdot 125\,\mathrm{N} = 5.1 \cdot 10^{-3}\,\mathrm{mm}^{-1}$ using

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} 2.04 & -0.12 & 0 & -2.55 & 0.27 & 0 \\ -0.12 & 4.43 & 0 & 0.27 & -12.1 & 0 \\ 0 & 0 & 38.0 & 0 & 0 & -65.2 \\ -2.55 & 0.27 & 0 & 10.2 & -1.07 & 0 \\ 0.27 & -12.1 & 0 & -1.07 & 48.4 & 0 \\ 0 & 0 & -65.2 & 0 & 0 & 260.9 \end{bmatrix} \cdot 10^{-5} \cdot \begin{Bmatrix} 300 \\ 0 \\ 0 \\ 125 \\ 0 \\ 0 \end{Bmatrix} .$$

The increase in $\widehat{EI}$ has no effect due to the transferred moment, similar to strains and stresses.

<u>Variant (c)</u> : loading acts at the top surface, also taken as reference plane

Given: Loading (Fig. 5),$\{n^0\} = (300,\, 0,\, 0)^T$ N/mm, $\{m\}$= $(200,\, 0, 0)^T$ N·mm/mm.

The laminate stiffness matrix (in MPa, mm), applying Eq. (3-18), reads

$$[K] = \begin{bmatrix} A & B \\ B & D \end{bmatrix} = \begin{bmatrix} 71392 & 2618 & 0 & 35696 & 1309 & 0 \\ 2618 & 71392 & 0 & 1309 & 35696 & 0 \\ 0 & 0 & 4600 & 0 & 0 & 2300 \\ 35696 & 1309 & 0 & 27675 & 873 & 0 \\ 1309 & 35696 & 0 & 873 & 19920 & 0 \\ 0 & 0 & 2300 & 0 & 0 & 1533 \end{bmatrix} ,$$

with [*A*] in $\mathrm{MPa \cdot mm}$, [*B*] in $\mathrm{MPa \cdot mm^2}$, [*D*] in $\mathrm{MPa \cdot mm^3}$.
Effective elastic constants: see variant (a). Effective laminate stiffnesses are:

$\widehat{EI}_x = 27675\,\mathrm{MPa \cdot mm^3}$, $\widehat{EI}_y = 19920\,\mathrm{MPa \cdot mm^3}$, $\widehat{GI}_t = 4 \cdot 1533\,\mathrm{MPa \cdot mm^3}$.:

According to Eq. (3-18) the strains of the reference plane are calculated as

$$\{\varepsilon^0\} = \begin{Bmatrix} 1.7 \\ 0.1 \\ 0 \end{Bmatrix} \cdot 10^{-3} , \qquad \{\kappa\} = \begin{Bmatrix} 5.1 \\ -0.5 \\ 0 \end{Bmatrix} \cdot 10^{-3} \cdot 1/\mathrm{mm} .$$

Herewith, the vector of strains at the boundary surfaces of the laminae can be determined with Eq. (3-12)

layer number	1	1	2	2	3	3	4	4
fiber angle	0°	0°	90°	90°	90°	90°	0°	0°
z in mm	0	0.25	0.25	0.5	0.5	0.75	0.75	1.0
ε_x in $10^{-3}\cdot$mm/mm	1.7	2.9	2.9	4.2	4.5	5.5	5.5	6.8
ε_y in $10^{-3}\cdot$mm/mm	0.1	0.0	0.0	-0.2	-0.2	-0.3	-0.3	-0.4
γ_{yx} in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0

With the application of Eqs. (3-21) follow the lamina strains and stresses in the four laminae

z in mm	0	0.25	0.25	0.5	0.5	0.75	0.75	1.0
$\varepsilon_{\|\|}$ in $10^{-3}\cdot$mm/mm	1.7	2.9	0	-0.2	-0.2	-0.3	5.5	6.8
$\varepsilon_{\perp}$ in $10^{-3}\cdot$mm/mm	0.1	0	2.9	4.2	4.2	5.5	-0.3	-0.4
$\gamma_{\perp\|\|}$ in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0
$\sigma_{\|\|}$ in MPa	221.5	391.3	5.0	-9.6	-9.6	-24.2	730.8	900.6
$\sigma_{\perp}$ in MPa	5.4	7.5	27.4	38.9	38.9	50.5	11.7	13.7
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	0	0	0	0

4.2 CFRP: Asymmetric Stack $[0/90/0/90]$, $t = 1.0\,\mathrm{mm}$, $t_k = 0.25\,\mathrm{mm}$

Just the main results of this "cross-ply"-laminate are summed up in this paragraph.

Task: Determination of stiffness matrix, lamina strains and stresses (computation is linear).

Given: Loading , action plane = mid-plane = reference plane
$\{n^0\} = (140,\,0,\,0)^T\,\mathrm{N/mm}$, $\{m\} = (0,\,10,\,0)^T\,\mathrm{N{\cdot}mm/mm}$;

Computation:

According to the asymmetric stack the sub-matrix $[B]$ is not zero anymore. The laminate stiffness matrix and its inverse read

$$[K] = \begin{bmatrix} 71392 & 2618 & 0 & -7755 & 0 & 0 \\ 2618 & 71392 & 0 & 0 & 7755 & 0 \\ 0 & 0 & 4600 & 0 & 0 & 0 \\ -7755 & 0 & 0 & 5949 & 218 & 0 \\ 0 & 7755 & 0 & 218 & 5949 & 0 \\ 0 & 0 & 0 & 0 & 0 & 383 \end{bmatrix},$$

$$[K]^{-1} = \begin{bmatrix} 1.63 & -0.06 & 0 & 2.13 & 0 & 0 \\ -0.06 & 1.63 & 0 & 0 & -2.13 & 0 \\ 0 & 0 & 21.7 & 0 & 0 & 0 \\ 2.13 & 0 & 0 & 19.6 & -0.72 & 0 \\ 0 & -2.13 & 0 & -0.72 & 19.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 261 \end{bmatrix} \cdot 10^{-5}.$$

As elastic constants are yielded

$$\widehat{E_x} = 71296\,\mathrm{MPa} = \widehat{E_y},\ \widehat{G_{xy}} = 4600\,\mathrm{MPa},\ \widehat{\nu_{xy}} = 0.037 = \widehat{\nu_{yx}}.$$

The strains in the reference plane are

$$\{\varepsilon^0\} = \begin{Bmatrix} 2.3 \\ -0.3 \\ 0 \end{Bmatrix} \cdot 10^{-3}, \qquad \{\kappa\} = \begin{Bmatrix} 2.9 \\ 2.0 \\ 0 \end{Bmatrix} \cdot 10^{-3} \cdot 1/\mathrm{mm}.$$

layer number	1	1	2	2	3	3	4	4
fiber angle	0°	0°	90°	90°	0°	0°	90°	90°
z in mm	−0.5	−0.25	−0.25	0	0	0.25	0.25	0.5
ε_x in $10^{-3}\cdot$mm/mm	0.8	1.6	1.6	2.3	2.3	3.0	3.0	3.7
ε_y in $10^{-3}\cdot$mm/mm	−1.3	-0.8	−0.8	- 0.3	−0.3	0.2	0.2	0.7
γ_{yx} in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0

z in mm	−0.5	−0.25	−0.25	0	0	0.25	0.25	0.5
$\varepsilon_{\|\|}$ in $10^{-3}\cdot$mm/mm	0.8	1.6	−0.8	−0.3	2.3	3.0	0.2	0.7
$\varepsilon_{\perp}$ in $10^{-3}\cdot$mm/mm	−1.3	-0.8	1.6	2.3	−0.3	0.2	3.0	3.7
$\gamma_{\perp\|\|}$ in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0
$\sigma_{\|\|}$ in MPa	108	206	-101	-33.6	304.5	403	33.7	101
$\sigma_{\perp}$ in MPa	-9.8	-3.3	12.5	20.6	3.2	9.7	28.7	36.8
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	0	0	0	0

4.3 CFRP: Stack $[0/10/90/70]$, $t = 0.5\,\mathrm{mm}$, $t_k = 0.125\,\mathrm{mm}$

Just the main results are summed up in this paragraph.

Task: Determination of stiffness matrix, lamina strains and stresses (computation is linear).

Given: loading, action plane = mid-plane = reference plane

$\{n^0\} = (100,\ 0,\ 0)^T\ \mathrm{N/mm}, \quad \{m\} = (15, 0,\ 0)^T\ \mathrm{N{\cdot}mm/mm}.$

Computation:

$$[Q']_1 = \begin{bmatrix} 133433 & 2618 & 0 \\ 2618 & 9351 & 0 \\ 0 & 0 & 4600 \end{bmatrix} \mathrm{MPa}, \quad [Q']_2 = \begin{bmatrix} 126207 & 6103 & 20183 \\ 6103 & 9608 & 1036 \\ 20183 & 1036 & 8084 \end{bmatrix},$$

$$[Q']_3 = \begin{bmatrix} 9351 & 2618 & 0 \\ 2618 & 133433 & 0 \\ 0 & 0 & 4600 \end{bmatrix} \mathrm{MPa}, \quad [Q']_4 = \begin{bmatrix} 11559 & 14926 & 5272 \\ 14926 & 106611 & 34607 \\ 5272 & 34607 & 16907 \end{bmatrix},$$

$$[K] = \begin{bmatrix} 35069 & 3283 & 3182 & -3769 & 261 & -34 \\ 3283 & 32376 & 4455 & 261 & 3247 & 803 \\ 3182 & 4455 & 4274 & -34 & 803 & 261 \\ -3769 & 261 & -34 & 749 & 86 & 37 \\ 261 & 3247 & 803 & 86 & 622 & 158 \\ -34 & 803 & 261 & 37 & 158 & 106 \end{bmatrix},$$

$$[K]^{-1} = \begin{bmatrix} 7.22 & -0.37 & -4.82 & 36.1 & -1.53 & 6.74 \\ -0.37 & 6.54 & -0.256 & -0.32 & -34.0 & 1.90 \\ -4.82 & -0.256 & 34.96 & -17.4 & -30.9 & -33.7 \\ 36.1 & -0.32 & -17.4 & 318.6 & -34.2 & -3.41 \\ -1.59 & -34.0 & -30.89 & -34.2 & 476 & -365 \\ 6.74 & 1.90 & -33.7 & -3.41 & -365 & 1559 \end{bmatrix} \cdot 10^{-5}.$$

Users may want elasticity constant values, although the pre-sumptions are violated:

$\widehat{E_x} = 65400\,\mathrm{MPa}$, $\widehat{E_y} = 55462\,\mathrm{MPa}$, $\widehat{G_{xy}} = 6893\,\mathrm{MPa}$, $\widehat{\nu_{xy}} = 0.001$, $\widehat{\nu_{yx}} = 0.001$.

Also, despite of the fact, that accurate effective laminate stiffnesses of a laminate (membrane loading) can be just computed for 'symmetric lay-up', they may be further interested in a stiffness information:

$\widehat{EI_x} = 749\,\mathrm{MPa}\cdot\mathrm{mm}^3$, $\widehat{EI_y} = 622\,\mathrm{MPa}\cdot\mathrm{mm}^3$, $\widehat{GI_t} = 4\cdot 106\,\mathrm{MPa}\cdot\mathrm{mm}^3$.

Strains and curvatures read:

$$\{\varepsilon^0\} = \begin{Bmatrix} 12.6 \\ -0.4 \\ -7.4 \end{Bmatrix} \cdot 10^{-3}, \qquad \{\kappa\} = \begin{Bmatrix} 83.9 \\ -6.7 \\ 6.2 \end{Bmatrix} \cdot 10^{-3} \cdot 1/\mathrm{mm}\,.$$

layer number	1	1	2	2	3	3	4	4
fiber angle	0°	0°	10°	10°	90°	90°	70°	70°
z in mm	−0.25	−0.125	−0.125	0	0	0.125	0.125	0.25
ε_x in $10^{-3}\cdot$mm/mm	-8.3	2.1	2.1	12.6	12.6	23.1	23.1	33.6
ε_y in $10^{-3}\cdot$mm/mm	1.3	0.4	0.4	-0.4	-0.4	-1.3	-1.3	-2.1
γ_{yx} in $10^{-3}\cdot$mm/mm	-9.0	-8.2	-8.2	-7.4	-7.4	-6.6	-6.6	-5.9

z in mm	−0.25	−0.125	−0.125	0	0	0.125	0.125	0.25
$\varepsilon_{\|\|}$ in $10^{-3}\cdot$mm/mm	-8.3	2.15	0.7	11.0	-0.4	-1.3	-0.5	0.2
$\varepsilon_{\perp}$ in $10^{-3}\cdot$mm/mm	1.3	0.4	1.9	1.25	12.6	23.1	22.4	31.3
$\gamma_{\perp\|\|}$ in $10^{-3}\cdot$mm/mm	-8.98	-8.2	-8.3	-11.4	7.4	6.6	-10.6	-18.5
$\sigma_{\|\|}$ in MPa	-1109	288	97.7	1467	-22.4	-107	-13.5	107.8
$\sigma_{\perp}$ in MPa	-9.98	9.61	19.4	40.43	117	213	208	293.3
$\tau_{\perp\|\|}$ in MPa	-41.3	-37.7	-38.2	-52.6	34.1	117	-48.7	-84.9

4.4 CFRP: Stack $[0/90/45/-45]_s$, $t = 1.0\,\mathrm{mm}$, $t_k = 0.125\,\mathrm{mm}$

Task: Investigation of the differences of lamina stress states caused by three unity load cases: Mechanical Loading, Thermal Stressing and Moisture Stressing with a final comparison. Note: there exists no hygrothermal *loading* such as with an external force. Stresses are caused if internal or external constraints become active and in case of non-constant hygrothermal fields.

Computation of stiffnesses: action plane = mid-plane = reference plane, s = 2 with $[Q]$ in MPa, and $\{\alpha_T\}$ in $10^{-6}\,\dfrac{\text{mm}}{\text{mm}\cdot\text{K}}$, $\{\alpha_M\}$ in $10^{-4}\,\dfrac{\text{mm}}{\text{mm}\cdot\%}$,

$$\{\alpha_T\}_k = \begin{Bmatrix} 0.23 \\ 29 \\ 0 \end{Bmatrix}, \quad \{\alpha_M\}_k = \begin{Bmatrix} 1 \\ 42 \\ 0 \end{Bmatrix}$$

$$[K] = \begin{bmatrix} A & B \\ B & D \end{bmatrix} = \begin{bmatrix} 56499 & 17512 & 0 & 0 & 0 & 0 \\ 17512 & 56499 & 0 & 0 & 0 & 0 \\ 0 & 0 & 19493 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7093 & 528 & 242 \\ 0 & 0 & 0 & 528 & 4185 & 242 \\ 0 & 0 & 0 & 242 & 242 & 694 \end{bmatrix},$$

$$[K]^{-1} = \begin{bmatrix} 1.96 & -0.61 & 0 & 0 & 0 & 0 \\ -0.61 & 1.96 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.13 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14.4 & -1.56 & -4.48 \\ 0 & 0 & 0 & -1.56 & 24.6 & -8.04 \\ 0 & 0 & 0 & -4.48 & -8.04 & 148.4 \end{bmatrix} \cdot 10^{-5},$$

$\widehat{E_x} = 51071\,\text{MPa} = \widehat{E_y},\ \widehat{G_{xy}} = 19493\,\text{MPa},\ \widehat{\nu_{xy}} = 0.31 = \widehat{\nu_{yx}}$ and as laminate stiffnesses per width under bending and torsion follow from Eq. (3-28)

$$\widehat{EI_x} = 7093 > \frac{100000}{14.4} = 6944\,\text{MPa}\cdot\text{mm}^3,\ \widehat{EI_y} = 4185\,\text{MPa}\cdot\text{mm}^3,\ \widehat{GI_t} = 4\cdot 694\,\text{MPa}\cdot\text{mm}^3.$$

(a) Mechanical Loading : $\{n^0\} = (100,\ 0,\ 0)^T$ N/mm, $\{m\} = (0,\ 0,\ 0)^T$ N·mm/mm.

$$\{\varepsilon^0\} = \begin{Bmatrix} 1.96 \\ -0.61 \\ 0 \end{Bmatrix} \cdot 10^{-3}, \qquad \{\kappa\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

layer number	1	1	2	2	3	3	4	4
fiber angle	0°	0°	90°	90°	45°	45°	-45°	-45°
z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
ε_x in $10^{-3}\cdot$mm/mm	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
ε_y in $10^{-3}\cdot$mm/mm	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6
γ_{yx} in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0

z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
$\varepsilon_{\|\|}$ in $10^{-3}\cdot$mm/mm	2.0	2.0	-0.6	-0.6	0.7	0.7	0.7	0.7
$\varepsilon_{\perp}$ in $10^{-3}\cdot$mm/mm	-0.6	-0.6	2.0	2.0	0.7	0.7	0.7	0.7
$\gamma_{\perp\|\|}$ in $10^{-3}\cdot$mm/mm	0	0	0	0	-2.6	-2.6	-2.6	-2.6
$\sigma_{\|\|}$ in MPa	260	260	-75.9	-75.9	91.9	91.9	91.9	91.9
$\sigma_{\perp}$ in MPa	-0.55	-0.55	16.7	16.7	8.09	8.09	8.09	8.09
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	-11.8	-11.8	11.8	11.8

(b) Thermal Stressing by a temperature decay : $\triangle T = -100\,\mathrm{K}$ (e.g. after curing)

$$\{\varepsilon^0\} = \begin{Bmatrix} -0.26 \\ -0.26 \\ 0 \end{Bmatrix} \cdot 10^{-3}\,, \qquad \{\kappa\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

layer number	1	1	2	2	3	3	4	4
fiber angle	0°	0°	90°	90°	45°	45°	-45°	-45°
z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
ε_x in $10^{-3}\cdot$mm/mm	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26
ε_y in $10^{-3}\cdot$mm/mm	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26
γ_{yx} in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0

z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
$\varepsilon_{\|\|}$ in $10^{-3}\cdot$mm/mm	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26
$\varepsilon_{\perp}$ in $10^{-3}\cdot$mm/mm	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26	-0.26
$\gamma_{\perp\|\|}$ in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0
$\sigma_{\|\|}$ in MPa	-24.1	-24.1	-24.1	-24.1	-24.1	-24.1	-24.1	-24.1
$\sigma_{\perp}$ in MPa	24.1	24.1	24.1	24.1	24.1	24.1	24.1	24.1
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	0	0	0	0

For hygro-thermal formulas, see HSB 37103-02.

(c) Stressing by a moisture uptake : $\triangle M = 1\,\%$ (e.g. after saturation in a fluid)

$$\{\varepsilon^0\} = \begin{Bmatrix} 0.43 \\ 0.43 \\ 0 \end{Bmatrix} \cdot 10^{-3}\,, \qquad \{\kappa\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

layer number	1	1	2	2	3	3	4	4
fiber angle	0°	0°	90°	90°	45°	45°	-45°	-45°
z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
ε_x in $10^{-3}\cdot$mm/mm	0.43	0.43	0.43	0.43	0.43	0.43	0.43	0.43
ε_y in $10^{-3}\cdot$mm/mm	0.43	0.43	0.43	0.43	0.43	0.43	0.43	0.43
γ_{yx} in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0

z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
$\varepsilon_{\|\|}$ in $10^{-3}\cdot$mm/mm	0.43	0.43	0.43	0.43	0.43	0.43	0.43	0.43
$\varepsilon_{\perp}$ in $10^{-3}\cdot$mm/mm	0.43	0.43	0.43	0.43	0.43	0.43	0.43	0.43
$\gamma_{\perp\|\|}$ in $10^{-3}\cdot$mm/mm	0	0	0	0	0	0	0	0
$\sigma_{\|\|}$ in MPa	34.4	34.4	34.4	34.4	34.4	34.4	34.4	34.4
$\sigma_{\perp}$ in MPa	-34.4	-34.4	-34.4	-34.4	-34.4	-34.4	-34.4	-34.4
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	0	0	0	0

Superposition of lamina stresses and assessment:

$$\{\varepsilon^0\} = \begin{Bmatrix} 2.13 \\ -0.43 \\ 0 \end{Bmatrix} \cdot 10^{-3}, \qquad \{\kappa\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

layer number	1	1	2	2	3	3	4	4
z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
fiber angle		0°		90°		45°		-45°
$\sigma_{\|\|}$ in MPa	270	270	-65.6	-65.6	102	102	102	102
$\sigma_{\perp}$ in MPa	-10.8	-10.8	6.5	6.57	-2.2	-2.2	-2.2	-2.2
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	-11.8	-11.8	11.8	11.8

Note: Residual stresses from moisture uptake reduce curing and mechanical stresses.

4.5 GFRP: Symmetric stack $[0/90/90/0] \equiv [0/90]_s$, $t = 1.0\,\text{mm}$, $t_k = 0.25\,\text{mm}$

Task: Determination of stiffness matrix, lamina strains and stresses (necessary for a later strength analysis). The computation is linear. Two variants of reference planes (a) and (b) are investigated.

Given: Loading midplane$\{n^0\} = (300,\,0,\,0)^T\,\text{N/mm}$, $\quad \{m\} = (50,\,0,\,0)^T\,\text{N·mm/mm}$.

Computation:

Using Eq. (3-5), with the properties above at first engineering 'constants' and stiffness (elasticity) matrix of the UD material are computed

$$[Q] = \begin{bmatrix} 46062 & 3192 & 0 \\ 3192 & 11821 & 0 \\ 0 & 0 & 4500 \end{bmatrix} \text{MPa}.$$

$$[Q']_2 = [Q']_3 = \begin{bmatrix} 11821 & 3192 & 0 \\ 3192 & 46062 & 0 \\ 0 & 0 & 4500 \end{bmatrix} \text{MPa}, [Q']_1 = [Q']_4 = \begin{bmatrix} 46062 & 3192 & 0 \\ 3192 & 11821 & 0 \\ 0 & 0 & 4500 \end{bmatrix} \text{MPa}.$$

The laminate stiffness matrix, applying Eq. (3-17), reads

$$[K] = \begin{bmatrix} 28941 & 3192 & 0 & 0 & 0 & 0 \\ 3192 & 28941 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3842 & 266 & 0 \\ 0 & 0 & 0 & 266 & 1342 & 0 \\ 0 & 0 & 0 & 0 & 0 & 375 \end{bmatrix},$$

with $[A]$ in $\mathrm{MPa \cdot mm}$, $[B]$ in $\mathrm{MPa \cdot mm^2}$, $[D]$ in $\mathrm{MPa \cdot mm^3}$ and its inverse is

$$[K]^{-1} = \begin{bmatrix} 3.5 & -0.39 & 0 & 0 & 0 & 0 \\ -0.39 & 3.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 22.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 29.2 & -5.78 & 0 \\ 0 & 0 & 0 & -5.78 & 75.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 267 \end{bmatrix} \cdot 10^{-5}.$$

The elasticity constants (of membrane) and the effective stiffnesses of the laminate are

$\widehat{E_x} = 28590\,\mathrm{MPa} = \widehat{E_y}$, $\widehat{G_{xy}} = 4500\,\mathrm{MPa}$, $\widehat{\nu_{xy}} = 0.11 = \widehat{\nu_{yx}}$

$\widehat{EI_x} = 3842 > \dfrac{100000}{29.2} = 3424\,\mathrm{MPa \cdot mm^3}$, $\widehat{EI_y} = 1342 = \dfrac{100000}{75.7} = 1342\,\mathrm{MPa \cdot mm^3}$, $\widehat{GI_t} = 4 \cdot 375\,\mathrm{MPa \cdot mm^3}$.

According to Eq. (3-18) the strains of the reference plane are calculated

$$\{\varepsilon^0\} = \begin{Bmatrix} 10.5 \\ -1.2 \\ 0 \end{Bmatrix} \cdot 10^{-3}, \qquad \{\kappa\} = \begin{Bmatrix} 14.6 \\ -2.90 \\ 0 \end{Bmatrix} \cdot 10^{-3} \cdot 1/\mathrm{mm}.$$

Herewith, the strains at the boundary surfaces of the laminae can be determined

layer number	1	1	2	2	3	3	4	4
z in mm	−0.5	−0.25	−0.25	0	0	0.25	0.25	0.5
fiber angle	0°	0°	90°	90°	90°	90°	0°	0°
ε_x in $10^{-3} \cdot$ mm/mm	3.2	6.8	6.8	10.5	10.5	14.1	14.1	17.8
ε_y in $10^{-3} \cdot$ mm/mm	0.3	- 0.4	−0.4	- 1.2	−1.2	−1.9	- 1.9	2.6
γ_{yx} in $10^{-3} \cdot$ mm/mm	0	0	0	0	0	0	0	0

With the application of Eqs. (3-21) the lamina strains and stresses in the four laminae follow

z in mm	−0.5	−0.25	−0.25	0	0	0.25	0.25	0.5
$\varepsilon_{\|\|}$ in $10^{-3} \cdot$ mm/mm	3.2	6.8	-0.4	-1.2	-1.2	-1.9	14.1	17.8
$\varepsilon_{\perp}$ in $10^{-3} \cdot$ mm/mm	0.3	-0.4	6.8	10.5	10.5	14.1	-1.9	-2.6
$\gamma_{\perp\|\|}$ in $10^{-3} \cdot$ mm/mm	0	0	0	0	0	0	0	0
$\sigma_{\|\|}$ in MPa	148	314	1.8	-19.8	-19.8	-41.5	645	811
$\sigma_{\perp}$ in MPa	13.6	16.7	79.6	120	120	161	22.9	26.0
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	0	0	0	0

Note: The comparison with the CFRP example 4.1 shows that - according to minor anisotropy - the GFRP laminate strains diverge less.

4.6 GFRP: Stack $[0/90/45/-45]_s$, $t = 1.0\,\text{mm}$, $t_k = 0.125\,\text{mm}$

Task: Determination of lamina stresses for three unity load cases Mechanical Loading, Thermal Stressing and Moisture Stressing with a final comparison. Mid-plane is reference plane.

Given: Loading $\{n^0\} = (100,\,0,\,0)^T$ N/mm, $\{m\} = (0,\,0,\,0)^T$ N·mm/mm,s = 2

Computation: For hygro-thermal formulas, see HSB 37103-02 [6].

$$[K] = \begin{bmatrix} A & B \\ B & D \end{bmatrix} = \begin{bmatrix} 24754 & 7379 & 0 & 0 & 0 & 0 \\ 7379 & 24754 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8687 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2726 & 353 & 67 \\ 0 & 0 & 0 & 353 & 1923 & 67 \\ 0 & 0 & 0 & 67 & 67 & 462 \end{bmatrix},$$

$$\widehat{E_x} = 222554\,\text{MPa} = \widehat{E_y},\ \widehat{G_{xy}} = 8687\,\text{MPa},\ \widehat{\nu_{xy}} = 0.298 = \widehat{\nu_{yx}}.$$

$$\widehat{EI_x} = D_{11} = 27263 > D_{11}^{*-1} = 100000/37.7 = 2653\,\text{MPa}\cdot\text{mm}^3$$
$$\widehat{EI_y} = D_{22} = 1923 > D_{22}^{*-1} = 100000/53.5 = 1876\,\text{MPa}\cdot\text{mm}^3$$
$$\widehat{GI_t} = 4\cdot D_{66} = 4\cdot 462 > 4/D_{66}^{*-1} = 4\cdot 100000/218.1 = 4\cdot 459\,\text{MPa}\cdot\text{mm}^3 \quad .$$

$$\{\widehat{\alpha_T}\} = \begin{Bmatrix} 9.7 \\ 9.7 \\ 0 \end{Bmatrix} \{\widehat{\alpha_M}\} = \begin{Bmatrix} 12.8 \\ 12.8 \\ 0 \end{Bmatrix}$$

.

(a) Mechanical Loading : $n_x^0 = 100\,\text{N/mm}$ or $\sigma_x^0 = 100\,\text{MPa}$; results are symmetric

$$\{\varepsilon^0\} = \begin{Bmatrix} 4.43 \\ -1.3 \\ 0 \end{Bmatrix} \cdot 10^{-3}, \qquad \{\kappa\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

layer number	1	1	2	2	3	3	4	4
z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
fiber angle		0°		90°		45°		-45°
$\sigma_{\|\|}$ in MPa	200	200	-46.7	-46.7	76.6	76.6	76.6	76.6
$\sigma_\perp$ in MPa	-1.5	-1.5	48.2	48.2	23.4	23.4	23.4	23.4
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	-25.9	-25.9	25.9	25.9

(b) Temperature decay : $\triangle T = -100\,\text{K}$ (e.g. after curing)

$$\{\varepsilon^0\} = \begin{Bmatrix} -0.97 \\ -0.97 \\ 0 \end{Bmatrix} \cdot 10^{-3}\,, \qquad \{\kappa\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

layer number	1	1	2	2	3	3	4	4
z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
fiber angle		0°		90°		45°		-45°
$\sigma_{\|\|}$ in MPa	-13.3	-13.3	-13.3	-13.3	-13.3	-13.3	-13.3	-13.3
$\sigma_{\perp}$ in MPa	13.3	13.3	13.,3	13.3	13.,3	133	13.3	13.3
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	0	0	0	0

(c) Change of moisture concentration : $\triangle M = 1\,\%$ (e.g. after saturation in a fluid,[6])

$$\{\varepsilon^0\} = \begin{Bmatrix} 128 \\ 128 \\ 0 \end{Bmatrix} \cdot 10^{-3}\,, \qquad \{\kappa\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

layer number	1	1	2	2	3	3	4	4
z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
fiber angle	up	0°		90°		45°		-45°
$\sigma_{\|\|}$ in MPa	46	46	46	46	46	46	46	46
$\sigma_{\perp}$ in MPa	-46	-46	-46	-46	-46	-46	-46	-46
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	0	0	0	0

Superposition of lamina stresses and assessment

$$\{\varepsilon^0\} = \begin{Bmatrix} 4.74 \\ -1.01 \\ 0 \end{Bmatrix} \cdot 10^{-3}\,, \qquad \{\kappa\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

fiber angle		0°		90°		45°		-45°
z in mm	-0.5	-0.375	-0.375	-0.25	-0.25	-0.125	-0.125	0
layer number	1	1	2	2	3	3	4	4
$\sigma_{\|\|}$ in MPa	232	2132	-14	-14	109	109	109	109
$\sigma_{\perp}$ in MPa	-33.9	-33.9	15.8	15.8	-9.1	-9.1	-9.1	-9.1
$\tau_{\perp\|\|}$ in MPa	0	0	0	0	-26	-26	-26	-26

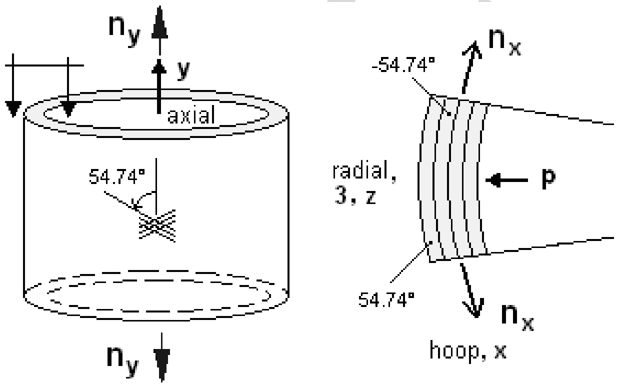
Note: Residual stresses from moisture uptake reduce curing stresses and mechanical stresses. This is different due to the different elastic and physical properties of GFRP and CFRP. But mind: the results presume that micro-cracking has not lead to a substantial change of the CTEs and CMEs.

4.7 Thin CFRP tube: Stack $[-54.7/54.7/-54.7/54.7]$, $t = 0.5\,\mathrm{mm}$, $t_k = 0.125\,\mathrm{mm}$.

In the case of sufficiently thin (approximately t/R <1/10, for details see HSB 37108-02), rotationally symmetrical FRP tubes (shells) CLT can be used, too. For the tube, the coupling effect disappears for reasons of symmetry and in-plane and laterally loaded plate problems can be considered separately.

Task: For the tube configuration (see sketch below), the stiffness matrix as well as the tube strains and lamina stresses are determined. The specimens are generally hoop wound tubes. This, usually makes, that the angle alters due to +-+- . Otherwise an effortful change of the winding direction would be necessary in order to reduce crossings in the winding pattern.

From reasons of rotational symmetry the curvature $\{\kappa\}$ is zero. y ≡axial, x ≡hoop (see Test Case 11 of WWFE-I, [16]), p = internal pressure value.



The fiber angle stems from application of the netting theory to an internally pressure-loaded water-hose where the "vessel formula" delivers $\tan 54.74 = \sqrt{2} = \sqrt{n_{hoop}/n_{axial}}$.

Given: Loading acts in mid-plane = reference plane

Applied loading $p \Rightarrow n_x = n_{axial} = p \cdot \pi \cdot \frac{r^2}{2}/(2 \cdot \pi \cdot r \cdot t), n_y = 2 \cdot n_x$,

$$\{n^0\} = (50,\ 100,\ 0)^T \text{N/mm}, \quad \{m\} = (0,\ 0,\ 0)^T\ \text{N·mm/mm};$$

Computation: Determination of stiffness matrices $[K] \Rightarrow [K]^{-1} \cdot 10^5$

$$\begin{bmatrix} 12114 & 14546 & 0 & 0 & 0 & 621 \\ 14546 & 32804 & 0 & 0 & 0 & 1207 \\ 0 & 0 & 15537 & 621 & 1207 & 0 \\ 0 & 0 & 621 & 252 & 303 & 0 \\ 0 & 0 & 1207 & 303 & 683 & 0 \\ 621 & 1207 & 0 & 0 & 0 & 324 \end{bmatrix} \Rightarrow \begin{bmatrix} 17.7 & -7.66 & 0 & 0 & 0 & -5.48 \\ -7.66 & 6.84 & 0 & 0 & 0 & -10.8 \\ 0 & 0 & 7.49 & -5.48 & -10.8 & 0 \\ 0 & 0 & -5.48 & 851 & -368 & 0 \\ 0 & 0 & -10.8 & -368 & 329 & 0 \\ -5.48 & -10.8 & 0 & 0 & 0 & 360 \end{bmatrix}$$

Effective elasticity constants, lamina strains and stresses are

$$\widehat{E_x} = \frac{A_{11} - A_{12}^2/A_{22}}{t} = \frac{11328}{0.5}\,\text{MPa},\ \widehat{E_y} = \frac{30674}{0.5}\,\text{MPa},\ \widehat{G_{xy}} = \frac{A_{66}}{t} = 31074\,\text{MPa},$$

$$\widehat{\nu_{xy}} = +\frac{A_{12}}{A_{22}} = \frac{14546}{32804} = 0.44 = -\frac{A_{12}^*}{A_{11}^*} - \frac{-7.66}{17.7},\ \widehat{\nu_{yx}} = (\widehat{E_y}/\widehat{E_x}) \cdot \widehat{\nu_{xy}} = 1.2,$$

$$\{\varepsilon^0\} = \begin{Bmatrix} 1.0 \\ 2.61 \\ 0 \end{Bmatrix} \cdot 10^{-3}, \qquad \{\kappa\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

layer number	1	1	2	2	3	3	4	4
z in mm	-0.25	-0.125	-0.125	-0	0	0.125	0.125	0.25
fiber angle	-54.7°	-54.7°	54.7°	54.7°	-54.7°	-54.7°	54.7°	54.7°
$\sigma_{\|\|}$ in MPa	260	260	260	260	260	260	260	260
$\sigma_{\perp}$ in MPa	19.8	19.8	19.8	19.8	19.8	19.8	19.8	19.8
$\tau_{\perp\|\|}$ in MPa	-7.0	-7.0	7.0	7.0	-7.0	-7.0	7.0	7.0

Note: The stresses in the laminae of this balanced 'water hose stack' are equal. According to linear CLT the axial strain is less than half of that in hoop direction. With increasing micro-crack damaging under higher loading both the strains approach the ratio 1:2 according to net-theory.

4.8 CFRP: $[0/90/45/-45]_S$, $t_{ks} \Leftrightarrow [0/90/45/-45]$, $t_{kn} = 2 \cdot t_{ks}$; $t = 1.0\text{mm}$

Task: Issues with stiffnesses of a symmetric versus a non-symmetric laminate (balanced laminate, material of example 4.4)

(1) Determination of stiffness matrices $[K] \Rightarrow [K]^{-1} \cdot 10^5$ and effective properties

* For the <u>*symmetric*</u> laminate $[0/90/45/-45]_S$ was computed (mid-plane is ref. plane)

$$\begin{bmatrix} 56499 & 17512 & 0 & 0 & 0 & 0 \\ 17512 & 56499 & 0 & 0 & 0 & 0 \\ 0 & 0 & 19493 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7093 & 528 & 242 \\ 0 & 0 & 0 & 528 & 4185 & 242 \\ 0 & 0 & 0 & 242 & 242 & 694 \end{bmatrix} \Rightarrow \begin{bmatrix} 1.96 & -0.61 & 0 & 0 & 0 & 0 \\ -0.61 & 1.96 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.13 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14.4 & -1.56 & -4.48 \\ 0 & 0 & 0 & -1.56 & 24.6 & -8.04 \\ 0 & 0 & 0 & -4.48 & -8.04 & 148.4 \end{bmatrix}$$

$\widehat{E_x} := 1/(1.96 \cdot 10^{-5} \cdot 1.00)\,\text{MPa} = 51071\,\text{MPa}$, or following Eq.(3-25) as before yields

$$\widehat{E_x} = 51071\,\text{MPa} = \widehat{E_y},\ \widehat{G_{xy}} = 19493\,\text{MPa},\ \widehat{\nu_{xy}} = 0.31 = \widehat{\nu_{yx}}$$

and as laminate stiffnesses per width under bending and torsion follow from Eq.(3-29)

$$\widehat{EI_x} = 7093\,\text{MPa} \cdot \text{mm}^3,\ \widehat{EI_y} = 4185\,\text{MPa} \cdot \text{mm}^3, \quad \widehat{GI_t} = 4 \cdot 694\,\text{MPa} \cdot \text{mm}^3.$$

<u>Notes</u>: (1) The A_{ij} are independent of the stack. (2) For a balanced laminate lay-up is $A_{16} = A_{26} = 0$ and $D_{16} = D_{26} = 0$. (3) Non-zero values $A_{16},\ A_{26}$ mean: coupling of extension and shear; non-zero $D_{16},\ D_{26}$ mean : coupled bending and twisting.

* For the <u>*non-symmetric*</u> laminate $[0/90/45/-45]$ follows (mid-plane is ref. plane, off-set to consider)

$$\begin{bmatrix} 56499 & 17512 & 0 & 7601 & -3723 & -1939 \\ 17512 & 56499 & 0 & -3723 & -154 & -1939 \\ 0 & 0 & 19493 & -1939 & -1939 & -3723 \\ 7601 & -3723 & -1939 & 6647 & 1459 & 969 \\ -3723 & -154 & -1939 & 1459 & 2769 & 969 \\ -1939 & -1939 & -3723 & 969 & 969 & 1624 \end{bmatrix} \Rightarrow \begin{bmatrix} 4.01 & -1.56 & 0.97 & -7.7 & 8.38 & 4.74 \\ -1.56 & 2.6 & 0.44 & 4.02 & -4.69 & 2.64 \\ 0.97 & 0.44 & 9.73 & -1.73 & 0.38 & 24.8 \\ -7.7 & 4.02 & -1.73 & 32.8 & -23.8 & -13.7 \\ 8.38 & -4.69 & 0.38 & -23.8 & 67.1 & -20.6 \\ 4.74 & 2.64 & 24.8 & -13.7 & -20.6 & 147.7 \end{bmatrix}$$

$$\widehat{E_x} = 51071\,\text{MPa} = \widehat{E_y},\ \widehat{G_{xy}} = 19493\,\text{MPa},\ \widehat{\nu_{xy}} = 0.31 = \widehat{\nu_{yx}}\ .$$

<u>Notes</u>: (1) The A_{ij} depend on the lay-up family, only. (2) The B_{ij} are zero for symmetric laminates. (3) Optimizing the stack is a means to reduce the B_{ij}-influence in the couplings which have an

impact on buckling and warping, on mounting of parts and joining. (4) The generation of material properties with test specimens - possessing non-zero B_{ij} - is not reliable, especially the generation of strength design allowables is questionable. (5) The bending-twisting affecting D_{ij} significantly depend on the stack. The D_{16}, D_{26} must be included in the structural analysis of non-symmetric stacks. (6) Following the CMH-17 as laminate stiffnesses per width under bending and torsion are recommended also for this non-symmetrc case and are to be taken from $[D^*]$:

$$\widehat{EI_x} = 1/D^*_{11} = 10^5/32.8\,\mathrm{MPa}\cdot\mathrm{mm}^3 = 3049\,\mathrm{MPa}\cdot\mathrm{mm}^3,$$

$$\widehat{EI_y} = 1/D^*_{22} = 10^5/67.1\,\mathrm{MPa}\cdot\mathrm{mm}^3 = 1490\,\mathrm{MPa}\cdot\mathrm{mm}^3.$$

(2) Investigation of a reference plane choice-independing bending stiffness matrix

Orthogonalization of the problem by $\left[\widetilde{D}\right] = [D] - [B]\cdot[A]^{-1}\cdot[B]$ leads to a bending stiffness matrix which is independent of the choice of the reference system. A comparison of the bending stiffness matrices

$$[D] = \begin{bmatrix} 6647 & 1459 & 969 \\ 1459 & 2769 & 969 \\ 969 & 969 & 1624 \end{bmatrix}, \text{ and } \left[\widetilde{D}\right] = \begin{bmatrix} 4708 & 1866 & 700 \\ 1886 & 2311 & 497 \\ 700 & 497 & 811 \end{bmatrix}$$

gives an information about the different stiffness element values of the former $[0/90/45/-45]$. The $D_{ii}$are smaller than the $\widetilde{D}_{ii}$. For a balanced symmetric laminate in the examples 4.1 (a) and (b) is obtained

- Mid-plane is reference plane:

$$\begin{bmatrix} 71392 & 2618 & 0 & 0 & 0 & 0 \\ 2618 & 71392 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4600 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9827 & 218.2 & 0 \\ 0 & 0 & 0 & 218.2 & 2072 & 0 \\ 0 & 0 & 0 & 0 & 0 & 383.3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1.40 & -0.05 & 0 & 0 & 0 & 0 \\ -0.05 & 1.40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 21.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10.2 & -1.07 & 0 \\ 0 & 0 & 0 & -1.07 & 48.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 260.9 \end{bmatrix}.$$

$$D = \begin{bmatrix} 9827 & 218.2 & 0 \\ 218.2 & 2072 & 0 \\ 0 & 0 & 383.3 \end{bmatrix} = D_{sym} \text{ and } \widetilde{D} = \begin{bmatrix} 9827 & 218 & 0 \\ 218 & 2072 & 0 \\ 0 & 0 & 383 \end{bmatrix}$$

- Top surface is reference plane:

$$\begin{bmatrix} 71392 & 2618 & 0 & 17848 & 655 & 0 \\ 2618 & 71392 & 0 & 655 & 17848 & 0 \\ 0 & 0 & 4600 & 0 & 0 & 1150 \\ 17848 & 655 & 0 & 14289 & 382 & 0 \\ 655 & 17848 & 0 & 382 & 6534 & 0 \\ 0 & 0 & 1150 & 0 & 0 & 671 \end{bmatrix} \Rightarrow \begin{bmatrix} 2.04 & -0.12 & 0 & -2.55 & 0.27 & 0 \\ -0.12 & 4.43 & 0 & 0.27 & -12.1 & 0 \\ 0 & 0 & 38.0 & 0 & 0 & -65.2 \\ -2.55 & 0.27 & 0 & 10.2 & -1.07 & 0 \\ 0.27 & -12.1 & 0 & -1.07 & 48.4 & 0 \\ 0 & 0 & -65.2 & 0 & 0 & 260.9 \end{bmatrix}$$

$$D = \begin{bmatrix} 14289 & 382 & 0 \\ 382 & 6534 & 0 \\ 0 & 0 & 671 \end{bmatrix} \text{ and } \widetilde{D} = \begin{bmatrix} 9827 & 218 & 0 \\ 218 & 2072 & 0 \\ 0 & 0 & 383 \end{bmatrix} = D_{sym}.$$

.

Note :Kommentar zum Bedeutung des Größenunterschieds und zu den Grenzen wäre schön. Wer weiß etwas???

5 Application Hints

A flow chart of the CLT design analysis is presented in Fig. 6. There it is shown that a stiffness matrix can be obtained on three different ways. The chosen way depends on the input and which accuracy is required. Carefully determined and evaluated experimental data are necessary as input properties. The calculation procedure is performed according to the posed task.

(1) Application remarks

- All the relationships above are practically the same as for the isotropic plate. They mathematically describe the behavior of the "thin" laminated plate, elastic theory is used.
- In a general lay-up (stack) case, an elastic plate-parallel symmetry plane does not exist which would enable to solve the membrane plate problem and the bending plate problem independently. This is only possible in case of a symmetrical lay-up where $[B] = 0$.
- The transversal shear moduli $\widehat{G_{xz}}$, $\widehat{G_{yz}}$ of most of the laminates are relatively small compared with isotropic plate materials. Therefore, it is only permitted to neglect the influence of the transversal (or interlaminar) shear stresses τ_{xz}, τ_{xz} or of the corresponding strains on the deformation behavior of the laminate if the slenderness ratio (length$\ell/$thickness t) is larger than a certain limit, a limit which depends on the grade of anisotropy. For instance, for the example: $\ell/t = 20$ and $\widehat{E_x}/\widehat{G_{xz}} = 50$, the deformation w^0 derived with the relationships above is 20% too low. In this context it is referred to the respective HSB sheet.
- The elasticity properties for the computation input are obtained from so-called "isolated" test specimens (weakest link failure behavior) whereas an embedded lamina underlies a redundant in-situ behavior. The embedded lamina will not lose its full stiffness after the first inter-fiber-crack. This is to consider when applying inelastic theory.
- For the reduction of micro-cracking from the edge effect the change of the orientation angle should be minimal, i.e. 45°. This will be possible if multi-directional tapes are optimally stacked
- Benefit of applying thin plies: Using thin plies, early micro-cracking with onset of micro-delaminations is reduced. The conflict of the designer (desires thin layers) with manufacture (desires to cluster if the same direction of neighbouring layers is given, thick layers) will be smoothed due to the presently available multi-angle tapes. This makes it possible to improve stack optimisation in order to minimize the non-symmetry. Also the ply-drop problem is reduced. An optimal permutation of thin layers can even lead to a compressive interlaminar normal stress σ_3
- The reduction of bending-torsion coupling ($A_{16} = A_{26} = 0$) by the right stacking is mandatory for a well designed and optimally behaving laminate.

The lower part of the flow chart describes the laminate structural analysis. Thereby it is distinguished whether the structural analysis problem is a statically indeterminate one or a statically determined one. In the latter case, after including the boundary conditions of the structural laminate element, all interesting structural variables can be directly determined.

The bottom part is the verification of the laminate design by a strength analysis which is usually performed as a "ply-by-ply" failure analysis. Design verification has been achieved if the required *Margin of Safety = (load-defined) Reserve Factor - 1* has been met.

(2) Note on modelling textile composites

The use of the proposed theory in chapter 3 to model the behaviour of textile-reinforced composites can be recommended in some cases but is not free of drawbacks. In general, the procedure should be used for pre-design purposes only. When the CLT for laminates composed of UD laminas should be applied to textile-reinforced composites, the following steps need to be undertaken: (1) Theoretical decomposition of the textile composites into idealised UD laminas ((i)-UD layers). (2) Evaluation of experimentally determined stress-strain curves in different directions of the textile-reinforced composite.(3) Identification of the engineering constants of the i-UD layer by inverse identification or by numerical analysis.

- *Composites reinforced with non-crimp fabrics*
Composites reinforced with non-crimp fabrics (NCF) can usually be modelled without any major difficulties by the CLT for laminates composed of UD laminas as long as the fibre volume content in thickness direction ??? is low enough. Most of the NCF composites have a polymeric stitch thread which smelts during manufacturing. In this case, the CLT can be recommended. The correctness of CLT-based predictions for NCF composites with a reinforcing stitch thread decreases with increased stitch thread volume content density.

- *Composites reinforced with woven fabrics*
It has to be stated that the reverse identification of engineering constants for i-UD layers of woven composites is not always possible especially when the warp/fill(weft) fibre content is not 1:1 such as a plain weave (e.g. for atlas 1:4 reinforcement). Furthermore, research has shown that the reverse identification could lead to too high engineering constants so that the theoretically obtained stress-strain response of the woven composite is usually overestimated. To overcome this drawback, several researchers proposed so-called textile-specific correction factors. All engineering constants are multiplied with these factors in order to consider the influence of fibre waviness or of fibre misalignment on the engineering constants.

(3) Note on indexing of Poisson's ratios

The definition of the Poisson ratios is not standardized [Tsai, Composites Design]. In the early days indexing of Poisson's ratio followed 'location' before 'cause'. This makes more sense and -in addition- follows the convention for the load quantities. This is the reason why, after many discussions and extensive literature work of the VDI-working group, the VDI 2014 guideline still sticked to the 'old' sequencing ν_{21}= -$\underline{s_{21}/s_{11}}$for the larger Poisson's ratio. Tsai uses the same [5] 'old' suffix sequencing for the major Poisson's ratio as in the VDI 2014 , also opposite to the HSB sheets. This indexing coresponds to a column normalization which allows for a simple interpretation of uni-axial tests.

However, in the last three decades more and more $\nu_{12} = -s_{21}/s_{22}$ (sequence of indices looks less logic than above) has been used for the larger Poisson's ratio (row normalization), especially in the FE codes. This was backboned by the English literature. Of course, there is no objective reason to take ν_{21} or ν_{12}. This might have been the reason for the change. In the HSB ν_{12} is still used for the larger Poisson's ratio!

Therefore, the input numbers must be checked, always. In order to avoid misuse, each user of a program is asked to perform the Maxwell-Betti check (larger ν times smaller E-modulus equals smaller ν times larger E-modulus) to become sure with the input. This check works for the lamina and the laminate as well.

Eventually, it should be mentioned that the letter C is used for the spatial or 3D elasticity (stiffness) matrix and the letter S for the compliance matrix as the inverted formulation of elasticity matrix. This is just the other way round as the letters say and some authors have unfortunately mixed it up.

Acknowledgment

Mr. Minderhoud (Fokker) was the originator of issue A of this sheet.
Dr. B. Grüber, ILK (TUD) and Mr. A. Hauffe, ILR (TUD) is very much thanked for cross-checking the examples, for valuable comments, and for significant improvements of the contents. Eventually, the author is grateful to Prof. Dr. Stephen Tsai, Mr. S. Braeutigam, Prof. Dr. H. Rapp (UniBw Muenchen)), Dr. R. Boehm (TUD), Dr. C. Mittelstedt, and eventually to Dr. J. Broede, IASB, for final checking and 'polishing' of this issue.
For this new issue, various codes have been applied as not the full desired output could be provided by a single code: (1) For the linear analyses the codes AlfaLam (Advanced Layerwise Failure Analysis of Laminates from KLuB, TU-Darmstadt was applied; (2) program eLamX (linear) from ILR, TU-Dresden (TUD); (3) a company program at RUAG.

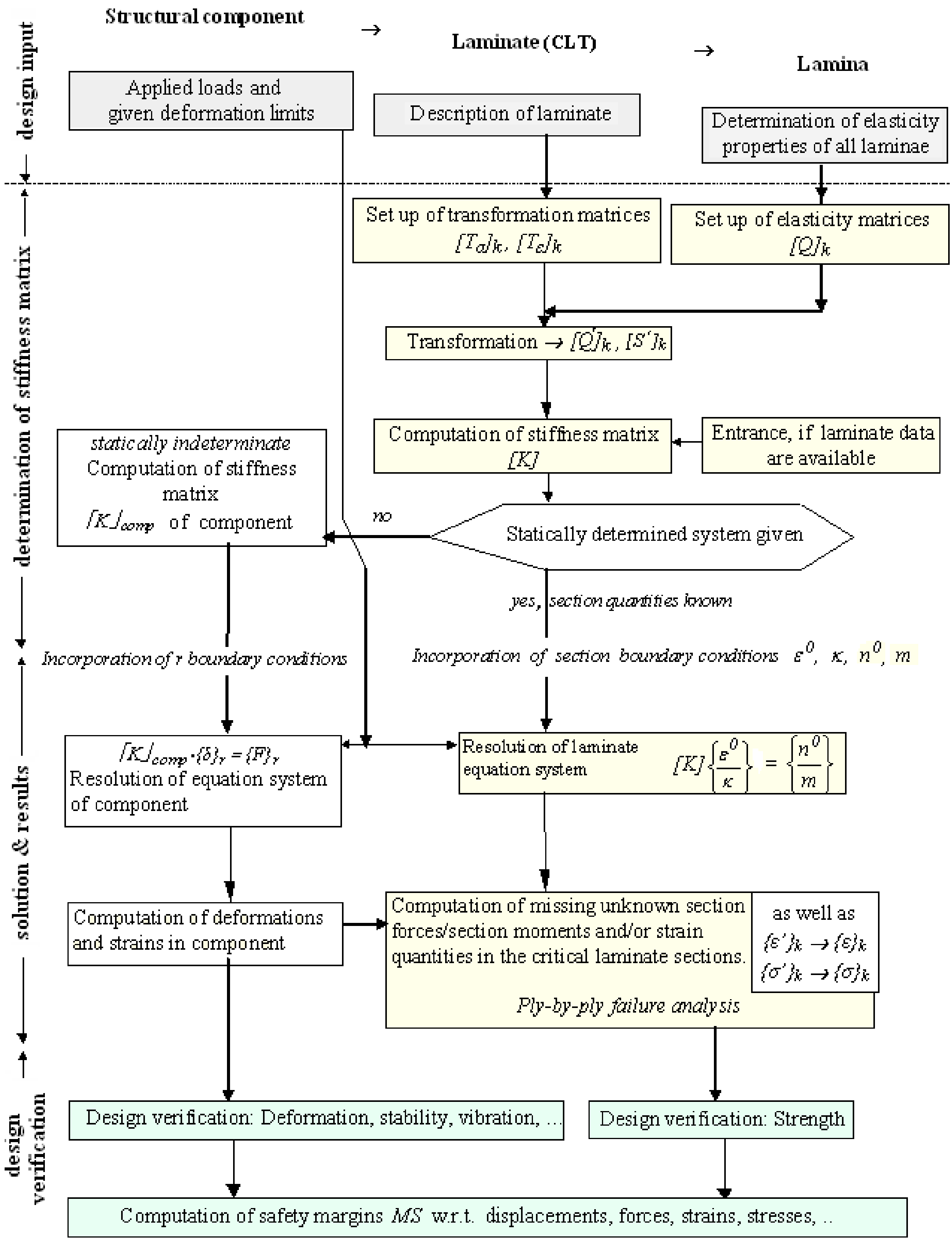


Figure 6: Flow chart of laminate analysis

6 Annexes

6.1 Consideration of offset of section forces

Section forces which do not act in the reference plane can be considered by

$$\{m\} = (m_x, m_y, m_{xy})^T = \int_{z_{bot}}^{z_{top}} \{\sigma'\} \cdot z \cdot dz - [z_{off}] \cdot \{n^0\} \quad \text{with} \tag{6-1}$$

$$[z_{off}] = \begin{bmatrix} z_{off_x} & 0 & 0 \\ 0 & z_{off_y} & 0 \\ 0 & 0 & z_{off_{xy}} \end{bmatrix}$$

defining the offset of the action plane of each section force. Values for each of the three section forces $\{n^0\}$ and three section moments $\{m\}$ are derived from component analysis. Of course, one could geometrically link those section forces and moments which act in the same plane.
Assuming a linear law of elasticity the following relations are obtained for the laminate

$$\left\{ \begin{array}{c} n^0 \\ m \end{array} \right\} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} - |z_{off}| \cdot \boldsymbol{A} \\ \boldsymbol{B} - |z_{off}| \cdot \boldsymbol{A} & \boldsymbol{D} - |z_{off}| \cdot \boldsymbol{B} \end{bmatrix} \cdot \left\{ \begin{array}{c} \varepsilon^0 \\ \kappa \end{array} \right\} = [K] \cdot \left\{ \begin{array}{c} \varepsilon^0 \\ \kappa \end{array} \right\}. \tag{6-2}$$

Usually, the loadings are delivered from the structural analyst such that just one z_{off} is to consider.

6.2 Non-compatible layer numbering and reference systems

with the z-coordinate downward and a layer numbering sequence increasing in downward z direction. This convention must be checked when using a finite element code and because the direction of the moments has to be considered in consequence.
Production will count differently: Usually, the bottom layer is the first produced layer and therefore termed layer 1 ('upward'-counting of layers due to the process). Further, the direction of z may be chosen upward or kept downward as in Fig. 3.
This very essential topic is addressed by an example in the Annex in order to generate a common understanding between designing and manufacturing engineers.

Intention of this Annex is to bridge and clarify the possible differences in understanding the lay-up and fiber orientation between the designing stress engineer, the manufacturing engineer and the software tool developer (FEA manuals). Further, different possibilities are available when specifying the reference COS and when defining ply numbering and ply orientation.
Practice demonstrates that applications of the classical lamination theory use different COS and different force/moment notation in comparison to the FEA software output. And, the FEA software can sometimes provide results (c), which do not match the classical lamination theory COS and force/moment notation used in this HSB sheet (b). Therefore it is mandatory that a check is performed to determine whether a common COS and a common ply orientation is used. An appropriate transformation is applied if that is not the case.

Also, a stress engineer is usually required to follow the manufacturer drawing (a) when building up his FE model.

In order to avoid any mistake, the following procedure is suggested for the laminate brick (a):

- Check of ply numbering and of orientation in the manufacturing document/drawing and comparison with the CLT definition
- Check of the loading direction and comparison with the CLT definition
- If different, transformation of layer angles necessary (i.e. complementary angle, different positive α counting), layer numbering and loading into the CLT reference system is required by the analysis defined in *Figs. 3, 4*).

For demonstrating this procedure an example is given and visualized in Fig. 7 below.

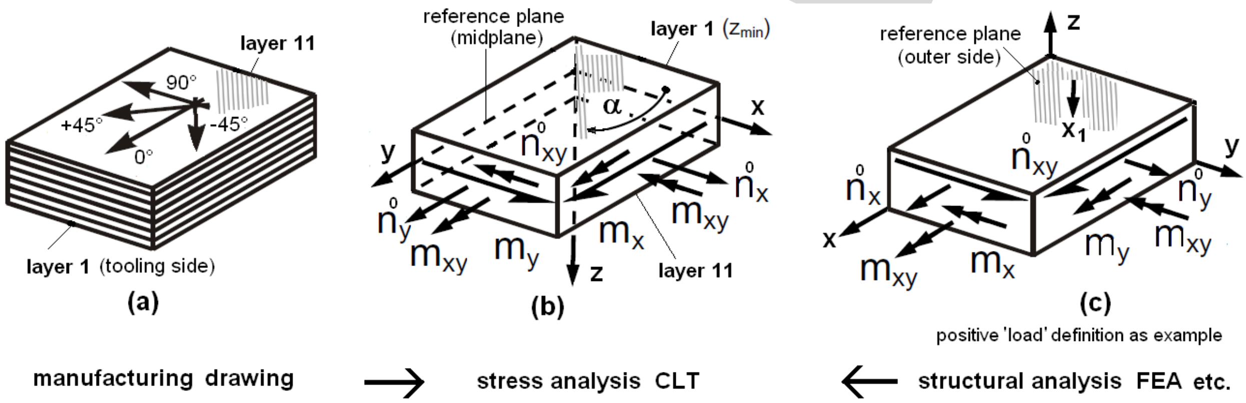


Figure 7: Visualization of non-compatible layer numbering

The different worlds which must be brought together are:

1. ***Manufacturing world,*** *sketch (a):* Given in the manufacturing drawing (a) is a lay-up [45/0/-45/90/0/0/90/-45/0/45/-45] with $t_k = 0.125$mm, $t = 1.375$mm with the outer surface (top layer) counting 11 with an orientation angle of -45°

2. ***CLT world****, sketch (b):* Reference plane = mid-plane= action plane. $\{n^0\} = (20,\ 40,\ 60)^T$N/mm, $\{m\} = (40,\ 15,\ 35)^T$N $\cdot$ mm/mm. According to the COS (b), which is applied in the stress analysis, the numbering and the positively angle-counted ($x \Rightarrow x_1$) lay-up is to be transformed into the chosen COS of (b) obtaining [45/-45/90/45/0/90/90/0/45/90/-45]. The top layer with an orientation angle of +45° counts 1, because the agreed sequence in writing the layers begins from left [45/0/..] beginning with 1 in positive direction of z

3. ***Analysis world***, FEA, sketch (c): Reference plane is bottom surface (layer 1). Further, the loading is transformed from (b) into (c). Due to the change in direction the forces and moments to be inserted in the analysis read:
$\{n^0\} = (40,\ 20,\ 60)^T$N/mm, $\{m\} = (15 - 40 \cdot t/2,\ 40{-}20 \cdot t/2,\ 35{-}60 \cdot t/2)^T=$

(-12.5, 26.25, 6.25)^{T}with $\{m\}$in N · mm/mm, $z_n = t/2$.
The lay-up will be denoted in (c) as [-45/90/45/0/90/90/0/45/90/-45/45]. The top layer counts 11 with an orientation angle of +45° ($x \Rightarrow x_1$)

4. ***Re-transformation*** of the FEA-results into the stress analysis documentation convention.

6.3 Miscellaneous Issues

(1) Material symmetries

Citing [5] the behaviour of anisotropic materials depends not as much on the number of independent elasticity properties as on the non-zero elements of the elasticity matrix. Shear stress-normal stress coupling are not coupled for UD and orthotropic materials. First, off-axis orientation

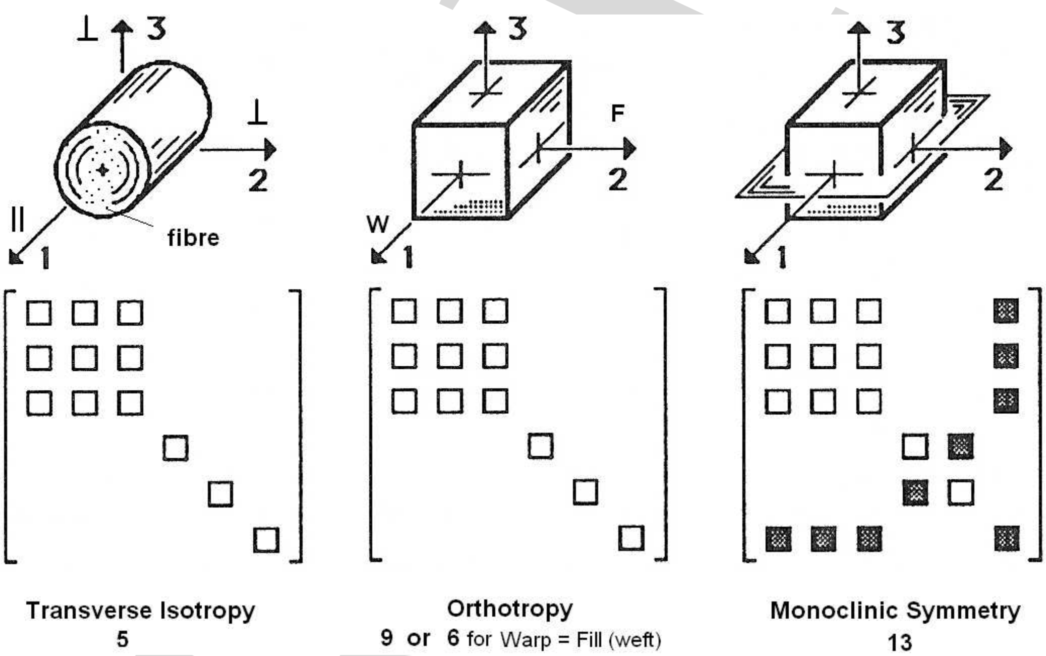


Figure 8: Occupation of elasticity matrix $[C]$in case of different material symmetry (after Tsai) with number of independent elasticity properties

(2) Influence of a varying angle on the elasticity properties

In Fig-9 three interesting properties are depicted as they vary over the angle.
The left sub-figure shows the decaying Young's modulus of a single lamina E_x and an angle-ply laminate $\widehat{E_x}$; the increasing shear moduli G_{xy}, $\widehat{G_{xy}}$; and the variation of Poisson's ratio ν_{xy}, $\widehat{\nu_{xy}}$.
From the $\widehat{\nu}_{xy}-$plot above can be concluded that Poisson's ratio may become much larger than the usual value of 0.3. This 'scissor'-effect is used in structural applications.

(3) Effects of the elements of the stiffness matrix $[K]$

The next figure shall visualize the effect of the elements (coefficients) of the stiffness matrix $[K]$of a transversely-isotropic UD material on the deformation of the laminate material element.

Note: Especially for stability analyses bending-torsion coupling is crucial.

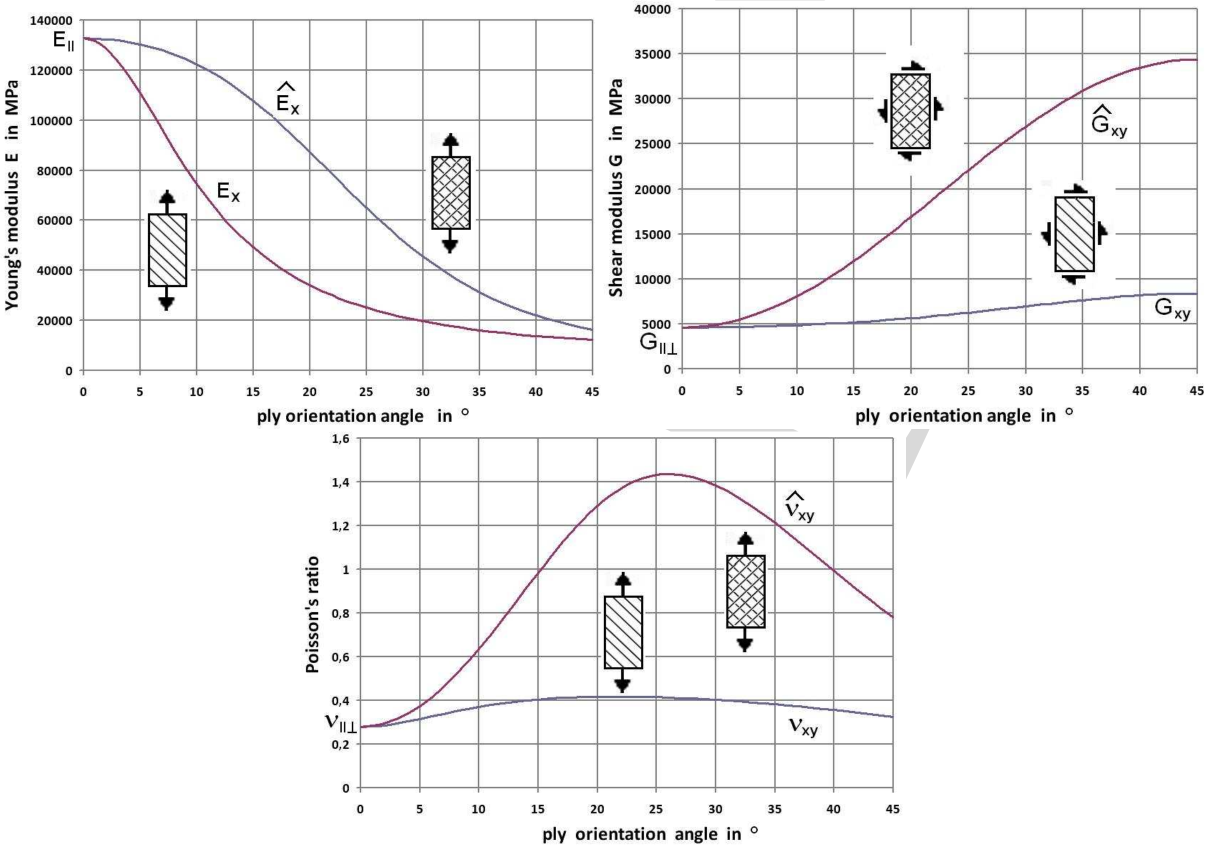


Figure 9: Dependence of the properties E_x, G_{xy}, ν_{xy} of the off-axis lamina and angle-ply on varying ply angle α and half-crossing angle ω

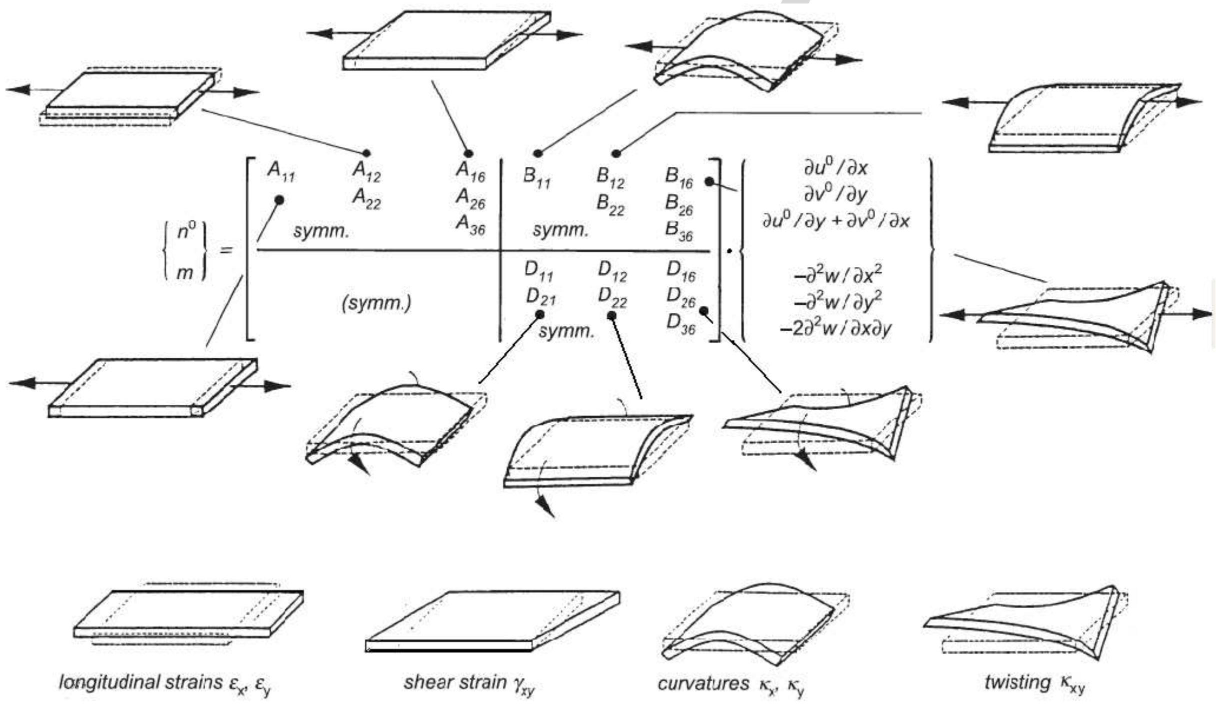


Figure 10: Effect of stiffness matrix elements on deformation of the laminate element

(4) 3D elasticity matrix of transversely-isotropic UD material

The spatial elasticity matrix $[C]$ in the column-normalized formulation is very complicated. Therefore for simplification, the elongation stiffnesses are abbreviated by the respective compliances which are related to the engineering constants known from Eq. (3-1). The equation reads column-normalized, whereby $C_{ij} = C_{ji}$,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = [C] \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{21} & C_{21} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{32} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \quad \text{(6-3)}$$

$$= \begin{bmatrix} S_{22}\cdot S_{33}-S_{32}^2 & S_{12}\cdot(S_{32}-S_{33}) & -S_{12}\cdot(S_{22}-S_{32}) & 0 & 0 & 0 \\ S_{12}\cdot(S_{32}-S_{33}) & S_{11}\cdot S_{33}-S_{12}^2 & -S_{11}\cdot S_{32}+S_{12}^2 & 0 & 0 & 0 \\ -S_{12}\cdot(S_{22}-S_{32}) & -S_{11}\cdot S_{32}+S_{12}^2 & S_{11}\cdot S_{22}-S_{12}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ \text{(symmetric)} & 0 & 0 & 0 & G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

with the denominator $N = S_{11}\cdot(S_{22}\cdot S_{33}-S_{32}^2) - S_{12}^2\cdot(S_{33}-2\cdot S_{32}+S_{22})$.

(5) The free-edge effect

The CLT is based on the assumption of a plane state of stress. However, in the vicinity of laminate joints, at ply drop-offs and at free laminate edges, at curved notches or at straight strips, plates or coupons, a 3D state of stress is faced. Fig.11 presents a general view of that together with some definitions. These stress states occur strongly localized with the possible consequence of an undesired premature failure associated by delamination.
Due to its underlying assumptions, CLT cannot capture such a 3D stress situation. However, the CLT is a good approximation in regions remote from the edge because the 3D stress fields at the free edges decay very fast, and in the inner unperturbed laminate regions CLT prevails. This causes that sizing of composite structures is often performed using CLT exclusively.
Nevertheless, the engineer who is working on composite laminated structures must have some knowledge about the nature of the free-edge effect and its implications.

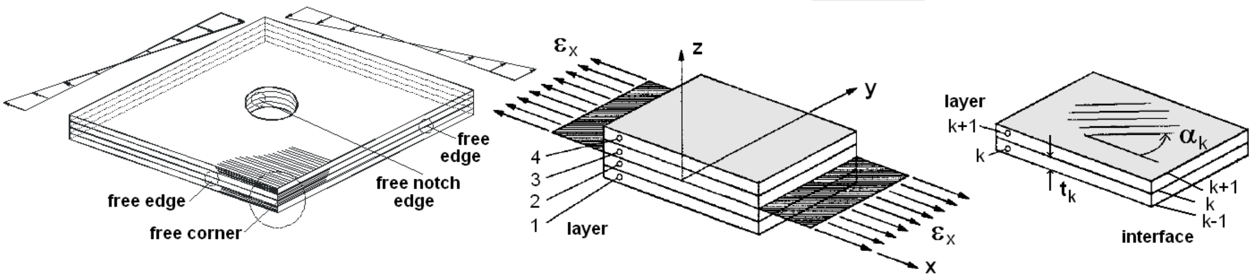


Figure 11: Sketch of a general plate strip under axial tension. Edge problems and definitions (courtesy Mittelstedt/Becker)

Free-edge effects are mainly caused by the generally differing elastic properties of adjacent laminate layers which results in a mismatch of the stress-strain-relationship and thus in an incompatible deformation behavior. Due to this incompatibility, a pronounced and potentially even singular 3D stress field is encountered at free edges of composite laminates at the interfaces between dissimilar layers which may be of substantial influence concerning the failure behaviour of such structures. This singularity is the result of the 'simple' linear analysis modelling. Note that these free-edge stress fields are usually confined to an area of the size of about one laminate thickness t.

The free-edge stress concentrations may generate edge-delaminations which cause an even higher stress singularity (stress intensity). These singularities are treated by means of fracture mechanics. The delamination effect is important in the assessment of test data from laminate coupons. Generally, the generated interlaminar stresses decrease if the difference between the orientations of two adjacent layers decreases.

The analysis of free-edge effects is rather difficult. Exact closed-form solutions do not exist, even for the simplest thinkable cases like plane symmetrically laminated specimens under uniaxial extension as depicted in the Figs. 12 and 13. An overview over the state of the art in this field can be gained with Ref. [22]. As a consequence, the analysis of free-edge effects is usually performed by using adequate finite element models that require a distinct mesh refinement in the vicinity of the free edges due to the stress singularities that are encountered at the free edges. However due to the

linear treatment the following result is faced: The finer the mesh the higher the singularity or the stress peak, respectively.
The free edge singularity cannot be assessed by strength criteria. However, it is nevertheless advantageous to alleviate the free edge effect of a laminated structural part qualitatively by comparing designs as far as possible despite of the fact that one cannot really quantify it.

Finally, the Figs. 12 and 13 illustrate the free-edge effect in a qualitative sense, indicating the significant interlaminar stresses, see Ref. [22]. For the laminate example [0/90/90/0], Fig.12 gives an idea about the distribution of the interlaminar shear stress τ_{yz} and of the tensile stress in thickness direction $\sigma_z \equiv \sigma_3$. Both these stresses act together and are responsible for micro-delaminations at the free edge. Fig.13 depicts the stress situation for the laminate example [45/-45/-45/45]. In both the cases y / t = 1 marks the end of the affected domain.

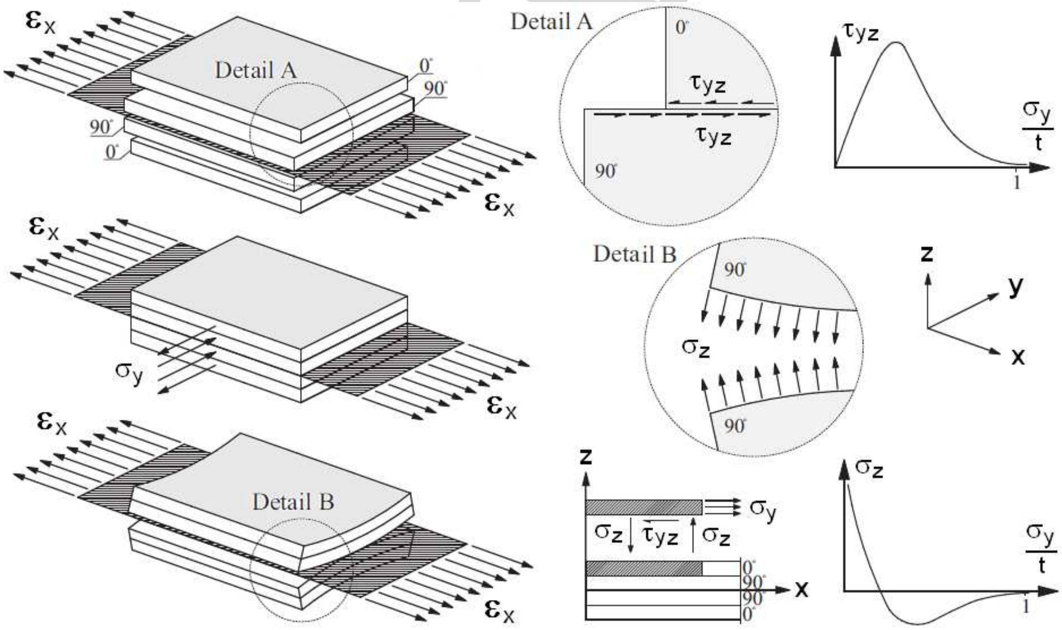


Figure 12: Plate strip showing the stresses under axial tension of a [0/90/90/0] laminate (courtesy Mittelstedt). Layer indices of σ_y are not indicated at the single ones

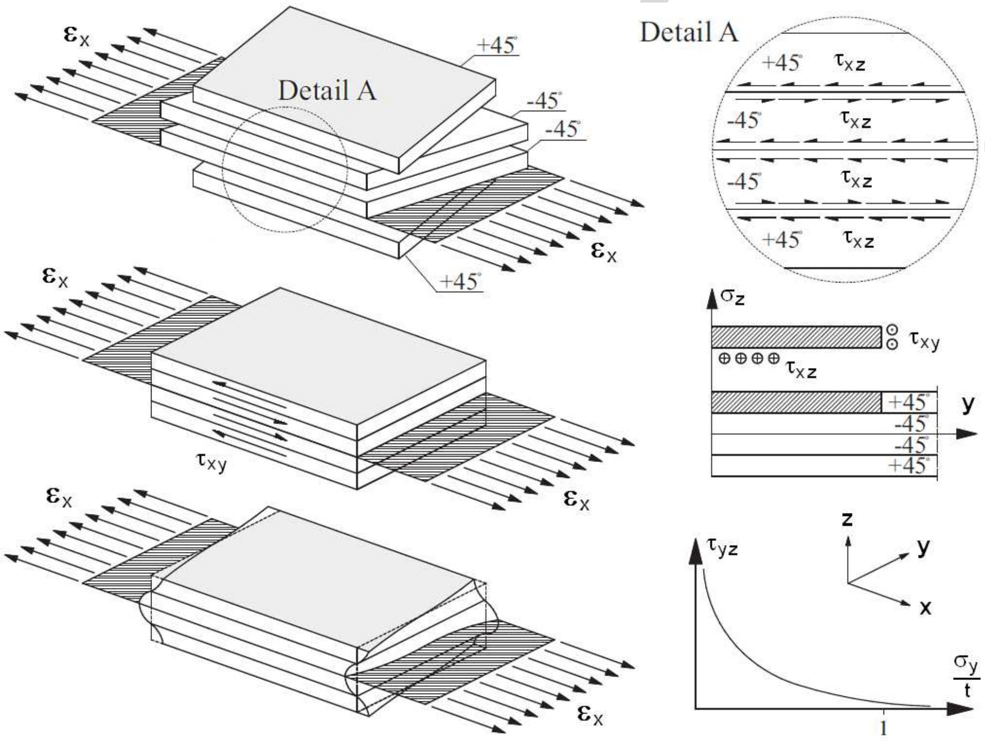


Figure 13: Plate strip showing the stresses under axial tension of a [45/-45/-45/45] laminate (courtesy Mittelstedt)

6.4 Visualization of plate deformations with derivation of the effective stiffness $\widehat{GI_t}$

Plate deformations and curvatures

Presumptions applied : CLT, linear theory, cross-sections remain planar, $D_{16} = D_{26} = 0$.

Bending**:** The two bending moments m_x and m_y cause the two w-deformations depicted in Fig.9, Subfigure (a). If the curvatures of the plate element are assumed to be constant then the w-deflection can be (index $_b$ for bending) formulated as

$$w_b(x, y) = -\int\int \kappa_x \cdot dx \cdot dx - \int\int \kappa_y \cdot dy \cdot dy = (\kappa_x \cdot x^2 + \kappa_y \cdot y^2)/2 \tag{6-4}$$

considering the boundary conditions $w_b(0,0) = \frac{\partial w_b}{\partial x}(0,0) = \frac{\partial w_b}{\partial y}(0,0) = 0$. Then, twisting curvature becomes $\kappa_{xy} = -2 \cdot \partial^2 w_b / (\partial x \cdot \partial y) = 0$.

Torsion**:** Like torsion in a thin-walled beam the twisting moment per width causes shear stresses in the plate varying over the wall thickness from a positive sign to a negative sign, or in other words change the direction over the thickness. This means that the parallelogram shape of a negative surface plane z < 0 turns by 90° for the other surface or in other words the shear deformations of the top and the bottom plane are opposite, Subfigure (d).
There are complementary shear stresses acting because the four torsion section (twisting) moments must act together to obtain equilibrium. Hence, four complementary twisting moments $m_{xy},\ m_{yx} =$

m_{xy} deform the plate element (e). The slope of a twisting plate changes in x-direction and y-direction as well, see Subfigures (b), (c). Subfigure (c) shows the deformation in a positive plane. The pure twisting deformation can be determined by

$$w_t(x,y) = -\int\int \frac{1}{2}\kappa_{xy} \cdot dx \cdot dy = -x \cdot y \cdot \frac{1}{2}\kappa_{xy} \tag{6-5}$$

considering as boundary conditions $w_t(0,0) = \frac{\partial w_t}{\partial x}(0,0) = \frac{\partial w_t}{\partial y}(0,0) = 0$. In consequence, for the bending curvatures is derived $\kappa_x = -\frac{\partial^2 w_t}{\partial x^2} = 0, \kappa_y = -\frac{\partial^2 w_t}{\partial y^2} = 0,$ q.e.d. Then, the deflection at the element corner ($y = dx$, $y = dy$) yields $w_t = -dx \cdot dy \cdot \kappa_{xy}/2$, see Subfigure (b).

The twisting curvature κ_{xy} causes shear strains in the planes $\gamma_{xy}(z) = z \cdot \kappa_{xy}$, varying linearly over the thickness. With the x-axis as twisting axis, $w' = \partial w/\partial x$ - as the slope of the surface- can be considered as negative twisting angle $-\vartheta$. Then, the curvature κ_{xy} represents the change of the twisting angle along the x-axis [18]

$$\kappa_{xy} = -2 \cdot \partial^2 w^0/(\partial x \cdot \partial y) = 2 \cdot \vartheta'. \tag{6-6}$$

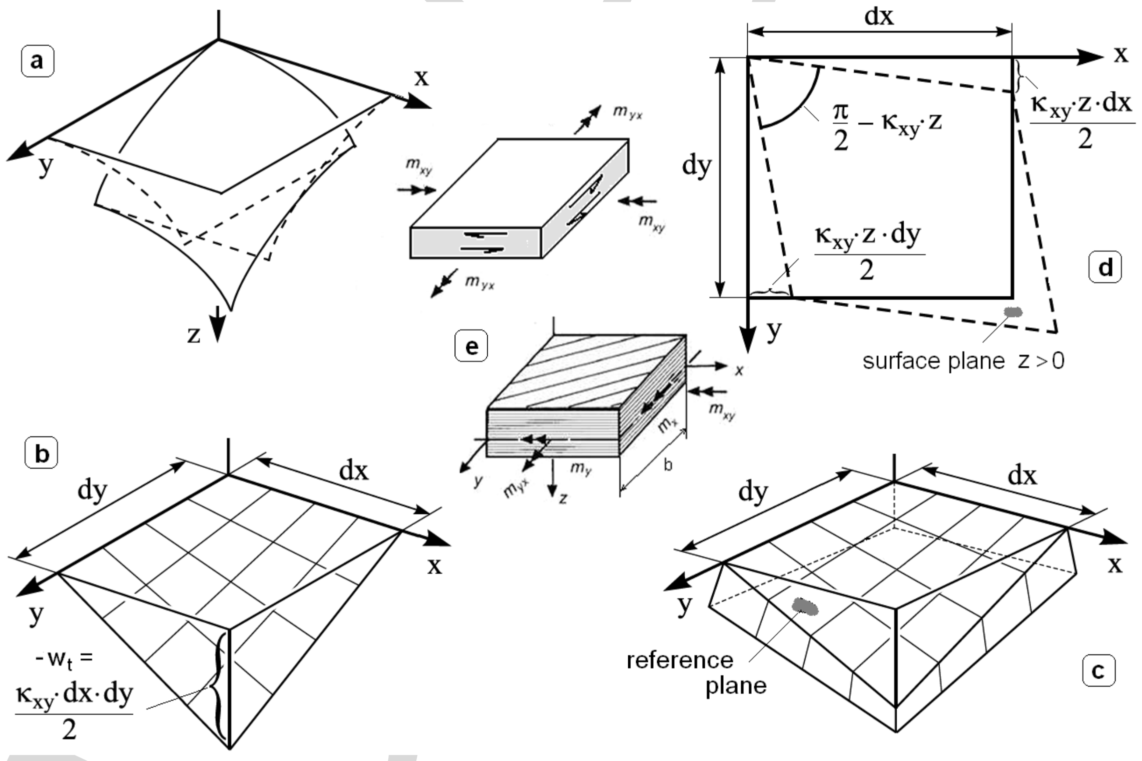


Figure 14: Visualization of plate deformations; (a) showing κ_x, κ_y due to solely bending moments m_x, m_y, (b) twisting κ_{xy} due to solely torsion moments, (c) w-deformation of the laminate, (d) shear straining in a distinct plane, (e) depicting loading situation

Derivation of the effective torsional stiffness $\widehat{GI_t}$

Considered is a wide strip with a narrow rectangle cross-section under a given twisting moment M_t. Just just shear stresses are envisaged. Then the required maximum torsional shear stress which determines dimensioning reads [20]

$$\tau_{xymax} = M_t/(b \cdot t^2/3). \tag{6-7}$$

Assuming τ_{xy} to be linearly distributed over the thickness $\tau_{xy}(z) = \tau_{max} \cdot z/(t/2)$

the integration of the shear stress over the thickness delivers

$$M_t^* = \frac{2}{t}\int_{-b/2}^{b/2}\int_{-t/2}^{t/2} \tau_{xy} \cdot z \cdot dA = \frac{2 \cdot \tau_{xymax}}{t}\int_{-b/2}^{b/2}\int_{-t/2}^{t/2} z \cdot z \cdot dy \cdot dz = \frac{2 \cdot \tau_{xymax}}{t} \cdot \frac{t^3 \cdot b}{3 \cdot 4}, \tag{6-8}$$

which demonstrates that $M_t^* = M_t/2$.
The missing half comes from far away minor shear stresses τ_{xz} which are not considered in the equation above and which act at the smaller slopes (the soap skin analogon says that a steeper slope outlines a higher stress) at the edges. This is made obvious, due to Fig. 12, by $dM_t = -(\tau_{xy} \cdot z + \tau_{xz} \cdot y)\, dy \cdot dz$ wherein the small stresses $\tau_{xz}$are multiplied by $y-$values which are much larger than the $z-$values.

In laminate plate theory, the torsional moment is defined as

$$-m_{xy} = \int_{-t/2}^{t/2} \tau_{xy} \cdot z \cdot dz \equiv M_t^*/b = M_t/2b. \tag{6-9}$$

Applying $\kappa_{xy} = 2 \cdot \vartheta'$, considering St. Venant Torsion $M_t = GI_t \cdot \vartheta'$, and $m_{xy} = D_{66} \cdot \kappa_{xy}$ (Eq. 3-28) the torsional stiffness becomes

$$GI_t = M_t/\vartheta' = -2 \cdot b \cdot m_{xy} \cdot 2/\kappa_{xy} = 4 \cdot b \cdot D_{66}. \tag{6-10}$$

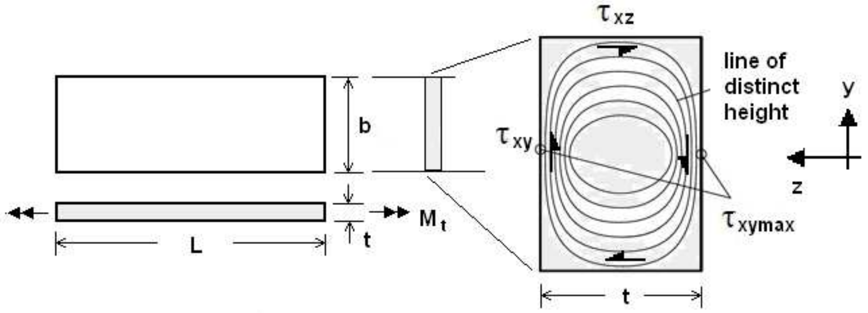


Figure 15: Projection of equidistant slope lines of the soap skin. Soap skin sketch in thickness direction exaggerated

Note on plate deformation, for comparison: In the central subfigure of Fig.13 is depicted how surface lines twist increasingly along the y-axis. If such a line at a position y has there the slope $\partial w/\partial x$, then the slope at y + $\triangle y$ is $\partial w/\partial x + \triangle y \cdot \partial(\partial w/\partial x)/\partial y$ [Kelly, Solid Mechanics, Part II]. During twisting the beam element rotates by a small angle $d\vartheta$. This angle has a relationship with the shear angle as

$$\gamma_{xy} = r \cdot d\vartheta/dx = r \cdot \vartheta'.$$

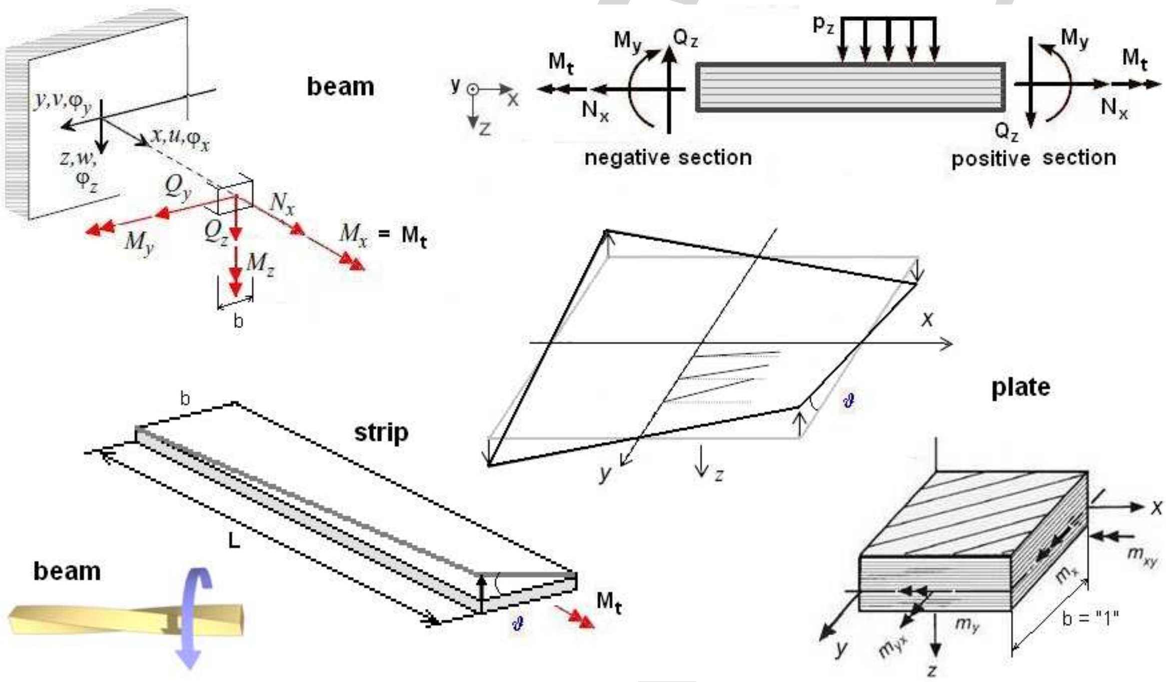


Figure 16: Definitions with a laminated plate and a laminated beam, see [HSB 01200-02]

Special case: laminated beam (b$\cong t$, compact), beam theory applied

Between plates and beams there is a significant difference: plates can twist which causes opposite shear stresses in the plane of the plate, varying over the wall thickness from a positive sign to a negative sign. Further, complementary shear stresses exist because 4 twisting moments must act together to obtain equilibrium. Therefore, the twist in the plate is caused due to all the 4 torsion (twisting) section moments m_{xy}, m_{yx}, see Fig.11.
With respect to a general beam case the full D-matrix is stressed. Assuming $m_y = 0$ yields

$$\begin{Bmatrix} m_x \\ 0 \\ m_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \cdot \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \qquad (6\text{-}11)$$

and in the coupled bending-twisting above the κ_y-curvature can be extracted. Hence, the equation system - due to $\kappa_y = -\kappa_x \cdot D_{12}/D_{22} - \kappa_{xy} \cdot D_{26}/D_{22}$ - reduces to

$$\begin{Bmatrix} m_x \\ m_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} - D_{12} \cdot D_{12}/D_{22} & D_{16} - D_{12} \cdot D_{26}/D_{22} \\ D_{16} - D_{26} \cdot D_{12}/D_{22} & D_{66} - D_{26} \cdot D_{26}/D_{22} \end{bmatrix} \cdot \begin{Bmatrix} \kappa_x \\ \kappa_{xy} \end{Bmatrix}.$$

When transferring the plate equations into beam ones, then bending section moment and torsion section moment are related according to the COS in Fig. 9 as

$$M_y = +b \cdot m_x \;\text{ and }\; M_t = -b \cdot 2 \cdot m_{xy}. \quad \text{(6-12)}$$

For small deflections, the curvatures can be related to the bending slope φ and the specific twist angle ϑ' using $\kappa_{\mathrm{x}} = -\varphi'_{\mathrm{x}} = -\mathrm{w}''$, $\kappa_{xy} = 2 \cdot \vartheta'$ which leads to the relationship,

$$\left\{ \begin{array}{c} M_y/b \\ M_t/2b \end{array} \right\} = \left[\begin{array}{cc} (D_{11} - D_{12}^2/D_{22}) & (D_{16} - D_{12} \cdot D_{26}/D_{22}) \\ (D_{16} - D_{26} \cdot D_{12}/D_{22}) & 2 \cdot (D_{66} - D_{26} \cdot D_{26}/D_{22}) \end{array} \right] \cdot \left\{ \begin{array}{c} \kappa_x \\ \kappa_{xy} \end{array} \right\} =$$

$$\left\{ \begin{array}{c} M_y \\ M_t \end{array} \right\} = \left[\begin{array}{cc} EI & K \\ K & GI_t \end{array} \right] \cdot \left\{ \begin{array}{c} -w'' \\ 2 \cdot \vartheta' \end{array} \right\} \quad \text{(6-13)}$$

with the effective beam stiffnesses (rigidities) for bending, bending-torsion coupling, torsion

$$\begin{array}{c} EI = b \cdot \widehat{EI} = b \cdot (D_{11} - D_{12}^2/D_{22}) \\ K = 2 \cdot b \cdot (D_{16} - D_{12} \cdot D_{26}/D_{22}) \\ GI_t = 4 \cdot b \cdot (D_{66} - D_{26} \cdot D_{26}/D_{22}) \end{array} . \quad \text{(6-14)}$$

For an orthotropic, symmetric stack $D_{16} = D_{26} = 0$.

<u>Isotropic case</u>: Rectangular beam of uniform cross-section along its length

$$EI = b \cdot (D_{11} - D_{12}^2/D_{22}) = b \cdot [Q_{11} \cdot \frac{t^3}{12} \cdot \left(1 - \nu^2\right)]\big) = b \cdot E \cdot t^3/12$$
$$GI_t = b \cdot 4 \cdot (D_{66}) = 4 \cdot b \cdot (E \cdot t^3/12) \cdot (1 - \nu)/2 \cdot (1 - \nu^2) = b \cdot G \cdot t^3/3.$$

GI_t has to be referred to the actual beam width according to HSB 32520-04 which delivers

$$GI_t = G \cdot I_t = G \cdot b \cdot t^3 \cdot \Phi_2 \quad \text{(6-15)}$$

with the shape factor Φ_2 from a Table 1 in HSB 32520-04: $\Phi_2 = 0.141$(b/t =1).
For a wide strip or plate $\Phi_2 = 1/3$(b/t =∞) and the torsional stiffness becomes $GI_t = G \cdot b \cdot t^3/3$, as still obtained.
Eventually, the angle of twist is calculated from the specific angle of twist applying

$$\theta = L \cdot \vartheta' = L \cdot M_t/(G \cdot I_t). \quad \text{(6-16)}$$

<u>Note:</u> Caused by the not existing curvature constraint in the width direction ($\kappa_y = 0$) the beam is somewhat less bending-stiff than the plate, $EI/b = b \cdot E \cdot t^3/12 < \widehat{EI} = E \cdot t^3/[12 \cdot \left(1 - \nu^2\right)]$.

Special case, laminated strip: (b>t), *uni-axial plate theory applied,* $m_y \neq 0,\ \kappa_y = 0$

For strips consisting of layers, the usability of the stiffening Poisson effect means that one either may use the plate equations or just the beam equation. This substantially depends on the real width b, on the stack with its fiber orientations and on the elasticity moduli.
For b≫t, *i*n consequence of the geometrical situation, one can assume cylindrical bending (a double curvature causes membrane strains which, however, are small from energetic reasons) which results in $\kappa_x \neq 0,\ \kappa_y = 0$. Hence it follows for the strip

$$b \cdot D_{11} > b \cdot \widehat{EI_x} > b \cdot (D_{11} - D_{12}^2/D_{22}). \tag{6-17}$$

For a narrow strip the plate equation may be seen an upper bound and the beam equation a lower bound. Poisson's ratios are usually of low effect $D_{12} \Rightarrow 0$ ($\nu = 0$ means, that the cross-section shape remains).

6.5 Reduction of Coupling and Mass Saving using Trace-normalized Stiffnesses

Appropriate stacking of a non-symmetric laminate helps to reduce the size of the [B] matrix. This is possible by using thin layers whereby coupling will be reduced and mass saving can be obtained, further. Modern tapes composed of non-crimp fabrics, representing sub-laminates with 2 or 3 fibre directions, allow now to use thin single layer stacks, whereby the coupling effects can be reduced (coupling does not basically mean low coupling stiffness values because these depend also on the chosen reference plane). The conflict between the designer who desires many thin layers (higher micro-cracking level) wants and the manufacturer who prefers fewer 'thick' layers for production cost reasons is not a bis issue anymore. Of course, to obtain an optimal stack in the sizing phase of the design requires the consideration of numerous permutations. These number of permutation can be reduced by applying in optimization the Trace-normalized stiffnesses proposed by Tsai. After optimisation, several sub-laminate stacks may be optimal and one has to decide which one should be taken.

Invariant elastic properties give a comprehensive information about the in-plane stiffness potential of a laminate consisting of a distinct composite material. All elastic stiffnesses are fractions of the trace invariant. The lamina (ply) and the laminate stiffness matrices read:

$$[Q]_k = \begin{bmatrix} \dfrac{E_1}{1-\nu_{21}\cdot\nu_{12}} & \dfrac{\nu_{12}\cdot E_2}{1-\nu_{21}\cdot\nu_{12}} & 0 \\ \dfrac{\nu_{21}\cdot E_1}{1-\nu_{21}\cdot\nu_{12}} & \dfrac{E_2}{1-\nu_{21}\cdot\nu_{12}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix},\ [A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}.$$

Conversion Tsai⇒HSB: $[a] \Rightarrow [A]^{-1}$,$[A*] \Rightarrow [A]/t$, ; $[a*] \Rightarrow [A*]^{-1}$). Viewing the formulations of Tsai, ν_{12} would be the smaller Poisson's ratio as in VDI 2014. Therefore, mind the course of subscripts and check according to Maxwell-Betti $\nu_{12}/E_1 = \nu_{21}/E_2$.

For the lamina and the laminate the traces are

$$Tr^{ply} = \sum(Q_{11} + Q_{22} + Q_{66}), \qquad Tr^{lam} = \sum(A_{11} + A_{22} + A_{66}).$$

Normalization follows by applying $Q_{ij}^{Tr} = Q_{ij}/Tr^{ply}$, $A_{ij}^{Tr} = A_{ij}/Tr^{lam}$.

	Q_{11}^{Tr}	Q_{22}^{Tr}	Q_{66}^{Tr}	$Tr^{ply}[GPa]$	$E_{\|\|}^{Tr}$	$E_{\perp}^{Tr}$	$G_{\|\|\perp}^{Tr}$	$\nu_{\|\|\perp}$
IM7/977 − 3	0.88	0.046	0.036	218	0.88	0.046		0.35
T800/Cytec	0.90	0.050	0.027	183	0.89	0.049		0.40
T700 C − ply	0.88	0.058	0.034	139	0.87	0.058		0.30
$AS4$/3501	0.86	0.056	0.044	162	0.85	0.055		0.30
IM6/epoxy	0.88	0.049	0.036	232	0.88	0.048		0.32
AS4/F937	0.89	0.058	0.027	168	0.88	0.057		0.30
T300/NS208	0.88	0.050	0.035	206	0.88	0.050		0.28
master	0.883	0.0502	0.0348	183	0.876	0.0500		0.300
cov	1.1%	0.44%	0.53%		1.2%	0.5%		4.1%

Figure 17: Normalized lamina quantities [courtesy S. Tsai]

One can firstly conclude that laminates usually have smaller coefficients of variation. Secondly, the normalisation leads to insensitivity among many laminae which justifies a master ply. Certification requires less tests due to accompanying invariant simulations. Tsai recommends as practical approach to reduce testing of large numbers of smooth coupons the testing of fewer open-hole specimens.

For deeper insight and further details about the practical application of the Trace method in design the reader is referred to literature from the originator of this idea, S. Tsai, which can be partly downloaded from *www.carbon-composites.eu/leistungsspektrum/fachinformationen.*

$[0/\pm 30]$	A_{11}^{Tr}	A_{22}^{Tr}	A_{66}^{Tr}	$Tr^{lam}[GPa]$	E_x^{Tr}	E_y^{Tr}	G_{xy}^{Tr}	ν_{xy}
IM7/977 − 3	0.65	0.091		218	0.52	0.072		1.2
T800/Cytec	0.66	0.091		183	0.50	0.069		1.3
T700 C − ply	0.64	0.099		139	0.52	0.079		1.1
$AS4$/3501	0.64	0.101		162	0.53	0.084		1.0
IM6/epoxy	0.65	0.093		232	0.52	0.074		1.2
AS4/F937	0.65	0.096		168	0.50	0.074		1.2
T300/NS208	0.65	0.093		206	0.52	0.075		1.2
master	0.647	0.0930		183	0.515	0.0745		1.18
cov	0.57%	0.36%	0.16%		1.0%	0.5%		8.4%
$[(0/\pm 30)_2/(90/\pm 60)]$								
IM7/977 − 3	0.46	0.28	0.13	218	0.42	0.25		0.40
T800/Cytec	0.47	0.28	0.13	183	0.42	0.25		0.43
T700 C − ply	0.46	0.28	0.13	139	0.42	0.25		0.40
$AS4$/3501	0.46	0.28	0.13	162	0.42	0.26		0.37
IM6/epoxy	0.46	0.28	0.13	232	0.42	0.25		0.39
AS4/F937	0.46	0.28	0.13	168	0.41	0.25		0.42
T300/NS208	0.46	0.28	0.13	206	0.42	0.25		0.39
master	0.463	0.278	0.130	183	0.418	0.252		0.398
cov	0.26%	0.12%	0.19%		0.18%	0.20%		1.75%
$[(0/\pm 45)_2/(\pm 45/90)]$								
IM7/977 − 3	0.38	0.29	0.16	218	0.32	0.24		0.48
T800/Cytec	0.39	0.29	0.16	183	0.31	0.23		0.52
T700 C − ply	0.38	0.29	0.16	139	0.31	0.24		0.49
$AS4$/3501	0.38	0.29	0.16	162	0.32	0.25		0.46
IM6/epoxy	0.39	0.29	0.16	232	0.32	0.24		0.48
AS4/F937	0.38	0.29	0.16	168	0.32	0.23		0.52
T300/NS208	0.39	0.29	0.16	206	0.316	0.24		0.48
master	0.385	0.293	0.161	183	0.5%	0.240		0.485
cov	0.08%	0.10%	0.09%			0.5%		2.2%

Figure 18: Normalized laminate quantities [courtesy S. Tsai]. The used non-crimp fabric consists of three layers with 3 fiber directions

7 Change Note

This extended version, Issue D, replaces Issue C dated January 11, 1978.