Summary

This sheet provides a guideline how coordinate systems may be chosen. For visualisation of the various topics, figures are presented.

Keywords: Coordinate systems, auxiliary system, path coordinate

References

[1] VDI 2014, German Guideline: Sheet 3 *Development of Fiber-Reinforced Plastic Components*. Beuth-Verlag, Berlin, 2006 (in German and English)

[2] HSB 40100-01: *Buckling - General remarks and terms*. Issue A, 2010

[3] Wiedemann, J.: *Leichtbau, Band 1 Elemente*. 2. Auflage, Springer, Berlin, 1996

[4] HSB 37103-01: *Classical Laminate Theory*. Issue D, 2010

[5] HSB 32520-03: *Beams with thin-walled cross sections loaded by shear forces acting at the shear center*. (not yet issued)

[6] HSB 32520-04: *St. Venant Torsion of beams with thin-walled cross sections - no warping.* (not yet issued)

[7] HSB 32520-05: *Torsion-loaded cross sections experiencing warping torsion.* (not yet issued)

[8] Gensichen, V. and Lumpe, G.: *Zur Leistungsfähigkeit, korrekten Anwendung und Kontrolle von EDV-Programmen für die Berechnung räumlicher Stabwerke im Stahlbau (Teil 1)*. Stahlbau 77 (6), 2008, 447-452.

1 General

In structure-mechanical analyses, the definition of the coordinate system (COS) is very differently performed. The same is valid for the path coordinate s in torsion and shear analysis. It is desirable to have a consistent use of the coordinate system x, y, z (deformations u, v, w) and of the path coordinate s as well.
This sheet provides some guidance how to do it.
It is further addressed whether it might be reasonable to 'fix' the coordinate system in the HSB or not.

Further information may be obtained from References not explicitly cited.

2 List of Symbols

Symbol	Unit	Description
m	N·mm/mm	section moment (moment per unit width of plate)
n	N/mm	section normal force (normal force per unit width)
p	N/mm	applied lateral distributed load (for beams)
p	N/mm², MPa	applied lateral distributed load, pressure (for plates)
q	N/mm	shear flow (for profile cross sections)
q	N/mm	section shear force or shear force per unit width (for plates)
s	mm	path coordinate
t	mm	plate thickness
$x,\ y,\ z$	mm	coordinates related to the elastic center (principal axes orientation)
$\overline{x},\ \overline{y},\ \overline{z}$	mm	coordinates related to the elastic center (arbitrary orientation)
$\widetilde{x},\ \widetilde{y},\ \widetilde{z}$	mm	coordinates related to chosen origin point O of auxiliary system
$x_{CG},\ y_{CG},\ z_{CG}$	mm	coordinates related to the center of gravity (arbitrary orientation)
$u,\ v,\ w$	mm	displacements in $x,\ y,\ z$ coordinate system
A	mm^2	cross section area
E	MPa	Young's modulus
EI	$\mathrm{N{\cdot}mm}^2$	bending stiffness of the beam (flexural rigidity)
M	N·m	bending moment and torque (for beams)
$N,\ Q$	N	normal force, shear force (for beams)
S	mm^3	static moment of cross section
S_E	N·mm	elastic (static) moment
S_ϱ	g	gravity (static) moment
κ	1/mm	curvature
ρ	mm	radius of curvature
ϱ	$\mathrm{g/mm}^3$	density
$\sigma_1,\ \sigma_2$	MPa	lamina (ply) normal stresses
$\tau,\ \tau_{21}$	MPa	shear stress, lamina (ply) shear stress

Superscripts, subscripts	Description
^{0}	denotes reference plane, mid-plane of a laminate
UD	uni-directional
$'$, $''$	derivations d/dx, d^2/dx^2
$_{CG}$	related to the center of gravity
$\sim$	tilde (auxiliary coordinate system)
$-$	bar over (related to the elastic center)
$_{x,y,z}$	related to principal coordinate axes
$\parallel$, $\perp$	parallel, perpendicular to the fiber direction

3 Analysis

3.1 Introduction

3.1.1 Conventions

1. Right-handed (right-turning screw) coordinate system (COS).
2. Positive section forces show at the positive section side (the normal on the cross section shows in positive direction) in the positive coordinate direction.
3. Positive moments are mathematically defined for a beam by a right hand rule.
4. Path coordinates in case of thin-walled beams basically take a continuous course. They should always start at a free profile edge where the shear flow is zero. Sub-path coordinates of profile off-branches should start at free edges as well.

For a beam (***Fig. 1***), no problem arises defining positive moments that are oriented in positive coordinate direction.

For the plate, there is no general definition possible without violating any of the several procedures used: Mathematically positive moments pointing in the direction of the positive coordinates, or the positive moment-curvature relation, or the following convention. Therein a general agreement is accepted: the double arrows of bending and twisting moments are defined in a way that in a reference zone (e.g. many civil engineers are accustomed to using a dashed line for the beam and zone for the plate) normal stress and shear stress point in the direction of the positive axis.

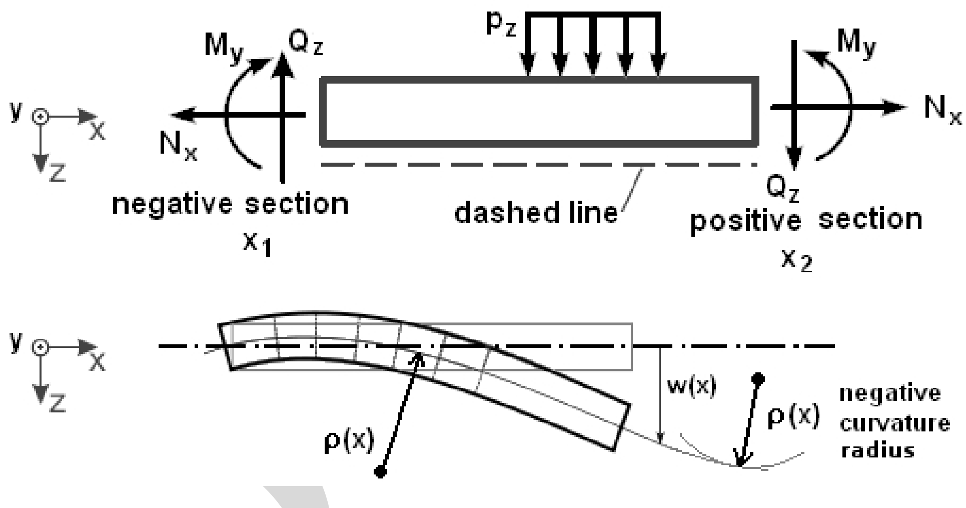

Figure 1: Coordinates, section forces and moment, deformation and curvature of a beam

The normal section force acts at the elastic center EC of the beam's cross section. In the case of homogeneity this becomes the center of gravity CG. The section moment is a free vector. The section shear force may act at the shear center.

3.1.2 Upward and downward coordinate system

Traditionally, a z-coordinate is understood to show upwards.

However, this 'feeling' is not often realized, for different reasons. Literature review shows that the z-coordinate of a beam or a plate is defined in both directions, upwards and downwards. For instance in the VDI 2014 (Ref. [1]), there is used a *downward* coordinate system and, because it is of advantage for the visualization and definition of the positive angle of fiber orientation, an *upward* coordinate system, also. These two different definitions are standard in composite engineering.

3.1.3 Curvature

From mathematics is known, that the curvature κ of the beam in *Fig. 1* is related to the function $w(x)$ by the relationship

$$\kappa = 1/\rho = w''/(1 + w'^2)^{3/2}. \qquad (3\text{-}1)$$

where ρ is the radius of the curvature. In geometrically linear elastic analysis the slope w' can be neglected in comparison to 1, leading to $\kappa \approx w''$. It is known from mechanics that the local curvature $\kappa(x)$ is proportional to the respective local bending moment $M_y(x)$.

3.2 Coordinate system

3.2.1 Beam

For a beam configuration - shown in the upper part of *Fig. 1* - the following relationships exist

$$N_x = \int \sigma_x \cdot dA, \qquad Q_z = \int \tau_{xz} \cdot dA, \qquad M_y = \int \sigma_x \cdot z \cdot dA, \qquad (3\text{-}2)$$

between the section quantities, with $dA = dy \cdot dz$ and integration over the total cross section area A.

As relationship between bending moment and curvature, it holds for a positively defined bending moment

$$M_y = -EI_y/\rho = -EI_y \cdot w''. \qquad (3\text{-}3)$$

As illustrated in the bottom part of *Fig. 1*, negative curvature is caused by a positive bending moment M_y.

In order to visualize the differences between beam and plate a 3D loaded beam is shown in ***Fig. 2***. The equilibrium of section quantities with the stresses requires the Eqs. (3-2) and

$$M_x = \int (\tau_{xz} \cdot y - \tau_{xy} \cdot z) \cdot dA, \quad Q_y = \int \tau_{xy} \cdot dA, \quad M_z = -\int \sigma_x \cdot y \cdot dA. \qquad (3\text{-}4)$$

Generally, the full set of 3D equilibrium equations for forces and moments of a beam reads

$$\frac{dN_x}{dx} + p_x = 0, \qquad \frac{dQ_y}{dx} + p_y = 0, \qquad \frac{dQ_z}{dx} + p_z = 0, \qquad (3\text{-}5)$$

$$\frac{dM_x}{dx} + m_x = 0, \qquad \frac{dM_y}{dx} - Q_z + m_y = 0, \qquad \frac{dM_z}{dx} + Q_y + m_z = 0, \qquad (3\text{-}6)$$

see *Fig. 2*, with p as distributed load and m as distributed moment.

The three moments M_x, M_y, M_z are defined to be positive in positive coordinate directions. Hence, in beam theory a positive curvature *(Fig. 2*, upper left part) causes a negative moment $M_y = -EI_y \cdot w''$ but a positive $M_z = +EI_z \cdot v''$.

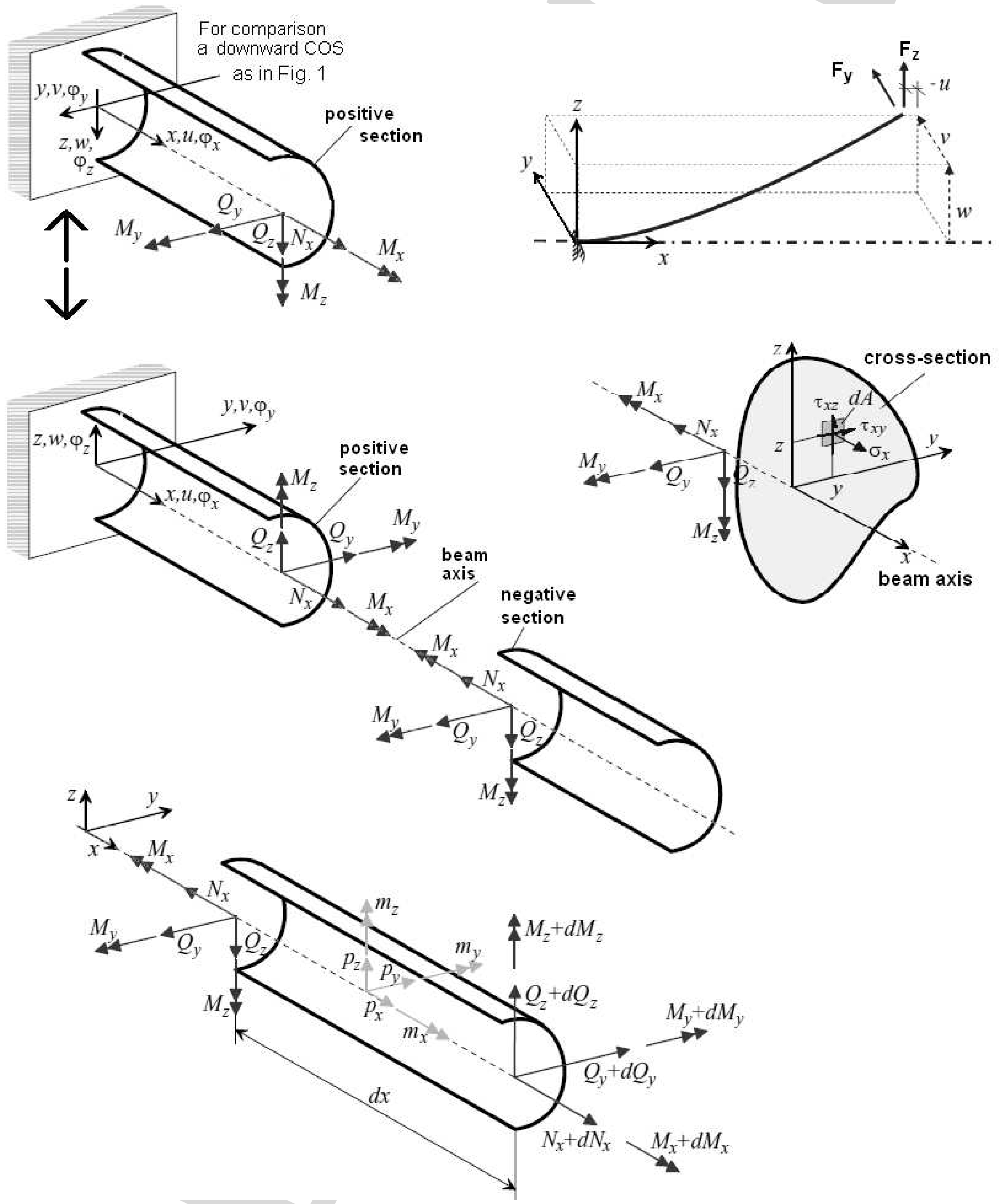

Figure 2: Coordinates, section quantities of a beam

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In order to show the identity of using an upward or a downward coordinate system, the upper left figure was introduced into *Fig. 2*. This figure demonstrates the general situation of the section quantities for the beam. Also, this 3D view outlines the differences to the 3D situation in the plate, shown in ***Fig. 3***.

3.2.2 Plate, Shell

Moments are not treated equally in beam and plate theory. In beam theory, the right hand rule applies for moments and M_y is a moment about the y-axis computed from σ_x stresses (*Figs. 1, 2*), whereas in plate theory m_y is a moment that is computed from σ_y-stresses and acts about the x-axis.

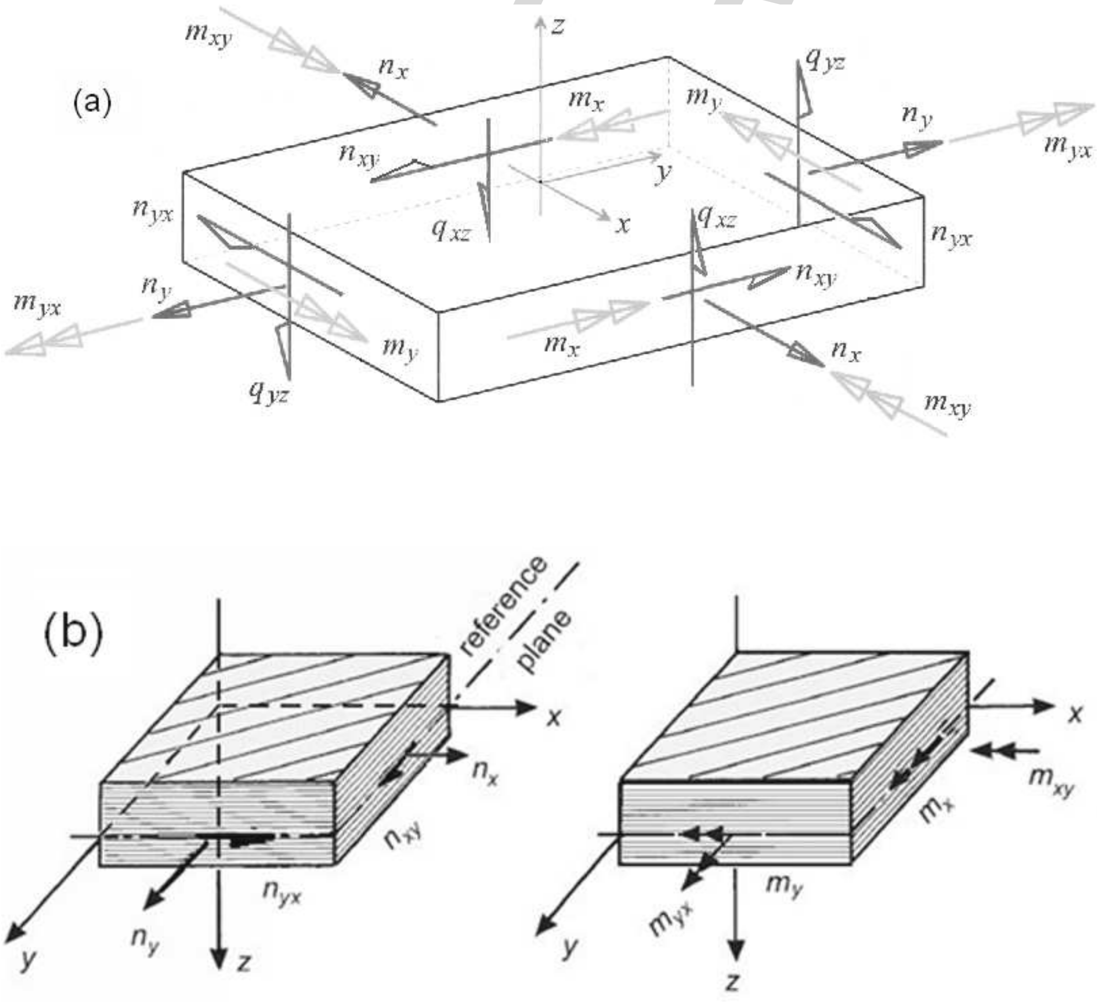


Figure 3: Coordinate system, section forces (stress resultants) and section moments of
(a) a plate (here, z upwards oriented) and
(b) a laminate (in laminate theory, z is usually downwards oriented)

The direction (see *Fig. 3*) of the double arrows of the bending and the twisting moments follows from the definition in Eqs. (3-7), (3-8) and (3-9). Positive moments always generate positive stresses at points with positive z-coordinates. From this definition follows that the double arrows may show into the direction of the coordinate system (m_x and m_{yx}) or against its direction (m_y and m_{xy}).

For a plate configuration, shown in the upper part of *Fig. 3*, the following relationships between the section quantities exist, applying $\tau_{yx} = \tau_{xy}$,

$$n_x = \int \sigma_x \cdot dz, \qquad q_x = \int \tau_{xz} \cdot dz \qquad m_x = \int \sigma_x \cdot z \cdot dz, \tag{3-7}$$

$$n_y = \int \sigma_y \cdot dz, \qquad q_y = \int \tau_{yz} \cdot dz, \qquad m_y = \int \sigma_y \cdot z \cdot dz, \tag{3-8}$$

$$n_{xy} = \int \tau_{xy} \cdot dz, \qquad m_{xy} = \int \tau_{xy} \cdot z \cdot dz, \tag{3-9}$$

and $n_{yx} = n_{xy}, \quad m_{yx} = m_{xy}$.

Of specific interest are relationships between bending and twisting moments and shear section forces

$$\frac{\partial m_x}{\partial x} + \frac{\partial m_{yx}}{\partial y} - q_{xz} = 0, \qquad \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} - q_{yz} = 0. \tag{3-10}$$

In the case of a laminated plate no neutral plane exists. Therefore often, the mid-plane is chosen as reference plane.

3.3 Fiber orientation angle and coordinate system of a uni-directional lamina (ply)

The orientation of the positive fiber angle α in ***Fig. 4*** is usually defined from x to $x_{||} \equiv x_1$. This affects the transformation of stresses and strains basically.

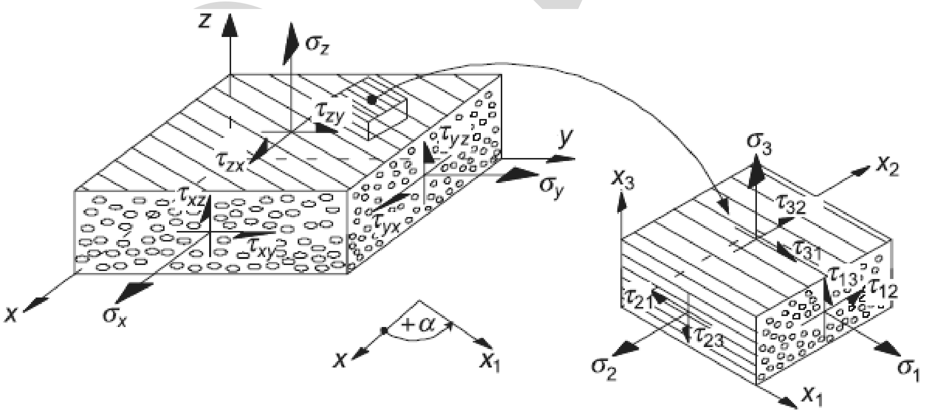


Figure 4: Coordinate systems, fiber orientation angle and stresses of a UD lamina, from Ref. [1]

As the opposite direction is used too - before doing any analysis - this error-prone point must be clarified.

Further, as already mentioned before, both, upward and downward coordinate systems are utilized. This has to be checked. The upward coordinate system corresponds to the steps in manufacturing. Then, number 1 is really the first layer placed.

3.4 Principal coordinate system, auxiliary systems and path coordinate

3.4.1 Static Moments, Center of Gravity, Elastic Center

Static moments of the cross section are defined as

$$S_{\tilde{y}} = \int_A \tilde{z} \cdot \mathrm{d}A, \quad S_{\tilde{z}} = \int_A \tilde{y} \cdot \mathrm{d}A \tag{3-11}$$

For a distinct coordinate system the static moments vanish. The origin of this distinct coordinate system is called 'Center of Area'.
The conditions $S_{\tilde{y}} = 0,\ S_{\tilde{z}} = 0$ can be used to determine the location of the Center of Area for a given cross section.

Gravity (static) moments of the cross section are defined as

$$S_{\varrho,\tilde{y}} = \int_A \varrho \cdot \tilde{z} \cdot \mathrm{d}A, \quad S_{\varrho,\tilde{z}} = \int_A \varrho \cdot \tilde{y} \cdot \mathrm{d}A \tag{3-12}$$

For the distinct coordinate system $x_{CG},\ y_{CG},\ z_{CG}$ the gravity (static) moments vanish. The origin of this distinct coordinate system is called 'Center of Gravity' (CG).
The conditions $S_{\varrho,y_{CG}} = 0,\ S_{\varrho,z_{CG}} = 0$ can be used to determine the location of the Center of Gravity for a given cross section.

Elastic (static) moments of the cross section are defined as

$$S_{E,\tilde{y}} = \int_A E \cdot \tilde{z} \cdot \mathrm{d}A, \quad S_{E,\tilde{z}} = \int_A E \cdot \tilde{y} \cdot \mathrm{d}A \tag{3-13}$$

For the distinct coordinate system $\overline{x},\ \overline{y},\ \overline{z}$ the elastic (static) moments vanish. The origin of this distinct coordinate system is called 'Elastic Center' (EC).
The conditions $S_{E,\overline{y}} = 0,\ S_{E,\overline{z}} = 0$ can be used to determine the location of the Elastic Center for a given cross section.

Note: Cross section normal forces acting at the Elastic Center do not cause bending moments or bending deformation, respectively.

Note: For homogeneous beams, Young's modulus $E(x, y, z)$ and density $\varrho(x, y, z)$ do not vary with $x,\ y$ and z, and therefore the Center of Area, the Elastic Center, and the Center of Gravity coincide.

3.4.2 Principal Axes

Bending stiffness quantities (related to the 2^{nd} moment of the cross section) are defined as

$$EI_{\tilde{y}} = \int_A E \cdot \tilde{z}^2 \cdot \mathrm{d}A, \quad EI_{\tilde{z}} = \int_A E \cdot \tilde{y}^2 \cdot \mathrm{d}A, \quad EI_{\tilde{y}\tilde{z}} = \int_A E \cdot \tilde{y} \cdot \tilde{z} \cdot \mathrm{d}A \tag{3-14}$$

For any point of the cross section, there exists a distinct system of perpendicular axes (which intersect at the respective point) for which the stiffness related to the product moment of the cross

section $EI_{\tilde{y}\tilde{z}}$ vanishes. These distinct axes are called 'Principal Axes', where principal axes intersecting each other in the Elastic Center are of particular interest. A coordinate system x, y, z with its origin in the Elastic Center and coordinate axes in line with the respective principal axes is called 'Principal Coordinate System'.
The condition $EI_{yz} = 0$ can be used to determine the directions of the principal coordinate system axes for a given cross section.

Note: Cross section bending moments acting about principal axes through the Elastic Center cause only bending deformation about the respective principal axis.

3.4.3 Shear Center

Cross section shear forces (lateral forces) may cause deflection (shear and bending deformation) as well as twist (torsional deformation). For a distinct point, the condition holds that the cross section shear force (when acting at this point) does not cause twist. This distinct point is called 'Shear Center' (SC).

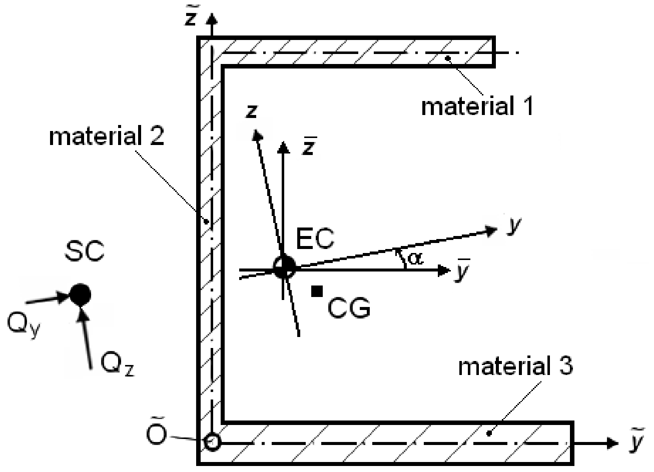
Note: When calculating shear quantities, the results obtained with the given formulas are valid only, if a principal coordinate system is used. General shear forces are decomposed into components in directions of the principal axes.

3.4.4 Principal and auxiliary coordinate systems

The origin of the principal coordinate system x, y, z is located in the elastic center EC of the cross section. The origin O of the auxiliary coordinate system (tilde system $\widetilde{x}$, $\tilde{y}$, $\tilde{z}$) is chosen so that numerical work is simplified, see ***Fig. 5***.

3.4.5 Path coordinate

The direction of the path coordinate s can be arbitrarily chosen. It usually orientates on the direction of a principal coordinate system and basically takes a continuous course. According to the profile's shape further sub-paths may be needed. For calculating shear stresses, continuous path and sub-paths should start at a free edge of the open cross section where the shear flow is known to be zero or at any point of the closed cross section (symmetry axis points are chosen always).
In the case of warping the procedure is different, see HSB 32520-05 (Ref. [7]).



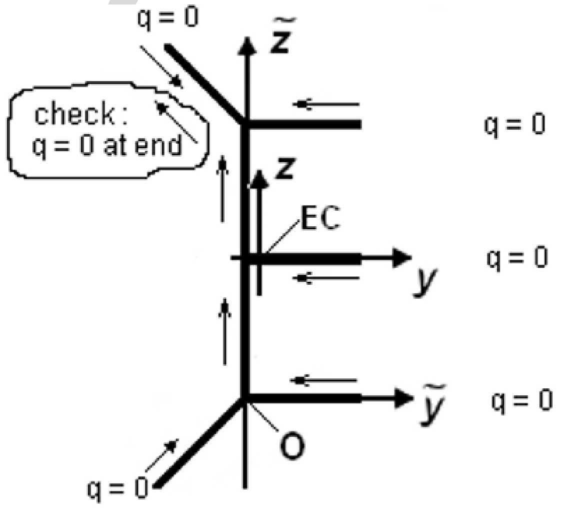


Figure 5: Principal and auxiliary (tilde and bar over) coordinate systems and path coordinate s. (EC means elastic center, CG means center of gravity, SC means shear center)

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