STRESS INTENSITY FACTORS AND ENERGY RELEASE RATE FOR ANISOTROPIC PLATES BASED ON THE CLASSICAL PLATE THEORY

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Abstract
The stress and displacement fields near the tip of a through crack in an anisotropic elastic plate are examined according to the classical plate theory. The plate is assumed to be subjected to bending moments, twisting moments, or transverse shear forces. In particular, the stresses at the prolongation of the crack and the relative crack face displacements are derived. The results are used to define stress intensity factors which are consistent with those for isotropic material. An explicit expression for the energy release rate in terms of the stress intensity factors is obtained using the path-independent J-integral. Analytic solutions are given for the stress intensity factors of a crack in an infinite plate under uniform bending moments, twisting moments, or transverse shear forces. It is shown that the stress intensity factors are zero and the stresses at the crack tip are finite if only twisting moments are applied. A universal relationship between the classical theory stress intensity factors and the Reissner theory stress intensity factors for thin orthotropic plates under symmetric bending is also derived.

1. Introduction

The stress and displacement fields near the tip of a through crack in a thin elastic plate under bending were first obtained by Williams [1] based on the classical plate theory. Sih, Paris and Erdogan [2] introduced two bending stress intensity factors to characterize the crack-tip fields. Analytic solutions of the stress intensity factors were derived for an infinite plate containing a finite crack subjected to uniform bending, twisting and shearing. Those due to twisting and shearing were later corrected by Zender and Hui [3].

The crack-tip stress field according to the classical plate theory is not entirely the same as that derived from the elasticity theory. The in-plane stresses in the classical plate theory exhibit the same $r^{-1/2}$ singularity, where r is the distance from the crack tip, as that predicted by the elasticity theory. However, the transverse shear stresses have a stronger $r^{-3/2}$ singularity. Moreover the angular distributions of stresses depend on Poisson's ratio. The discrepancy occurs because of the inability of the classical plate theory to satisfy the traction-free conditions for in-plane and transverse shear stresses independently at the crack faces. To remedy the inadequacy, the Reissner plate theory, which allows for transverse shear deformations, was used to obtain the crack-tip stress field that is consistent with that predicted by the elasticity theory [4]. Although the classical theory and the Reissner theory yield different crack tip fields, Simmonds and Duva [5] showed that the energy release rates calculated from the two theories are the same for thin plates. Invoking Simmonds and Duva's result, Hui and Zender [6] obtained a universal relationship between the classical theory stress intensity factors and

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the Reissner theory stress intensity factors for thin plates. Hui and Zender also showed that when correlating fracture toughness data for thin elastic plates it is sufficient and, in many cases, preferable to use the classical theory rather than the Reissner theory. This is because even a small amount of crack tip plasticity will render the Reissner theory fields invalid due to its small region of dominance relative to the plate thickness. The dominance of the classical plate theory is larger.

The aforementioned works are for isotropic plates and cannot be applied to treat composite plates, which are anisotropic. Similar works on anisotropic plates, however, appear to be limited. For the classical plate theory, Sih and Chen [7] provided the crack-tip stress field for anisotropic plates, containing parameters with rather complicated expressions. Neither the definition of stress intensity factors nor the energy release rate was given. It is the objective of this work to examine the crack-tip fields and extend the definition of stress intensity factors introduced in [2] to general anisotropic plates based on the classical plate theory.

In the following discussions the Latin indices range from 1 to 3, the Greek indices range from 1 to 2, summation over repeated indices is implied, and a comma in the subscript stands for partial differentiation.

2. Classical Plate Theory

In the classical plate theory, the displacements are assumed as

$$u_\alpha = x_3\phi_\alpha\left(x_1, x_2\right),\ x_3 = w\left(x_1, x_2\right), \tag{1}$$

where u_α are the in-plane displacements, w is the deflection, and ϕ_α are the rotations related to w by

$$\phi_\alpha = -w_{,\alpha}. \tag{2}$$

In the absence of transverse loads on the plate surfaces the equilibrium equations are

$$M_{\alpha\beta,\alpha} = Q_\beta,\ Q_{\beta,\beta} = 0, \tag{3}$$

where $M_{\alpha\beta}$ are the moments, Q_β are the transverses shear forces. From Eq. (3), the moments and shear forces can be expressed in terms of two stress functions, ψ_1 and ψ_2, as

$$M_{11} = -\psi_{1,2},\ M_{22} = \psi_{2,1},\ M_{12} = \frac{1}{2}\left(\psi_{1,1} - \psi_{2,2}\right), \tag{4}$$

$$Q_1 = -\eta_{,2},\ Q_2 = \eta_{,1},\ \eta = \frac{1}{2}\left(\psi_{1,1} + \psi_{2,2}\right). \tag{5}$$

It is assumed that $\sigma_{33} = 0$ and that the in-plane stresses $\sigma_{\alpha\beta}$ and transverse shear stresses $\sigma_{3\alpha}$ are given by

$$\sigma_{\alpha\beta} = \frac{12x_3}{h^3}M_{\alpha\beta}, \tag{6}$$

$$\sigma_{3\alpha} = \frac{3}{2h}\left(1 - \left(\frac{2x_3}{h}\right)^2\right)Q_\alpha, \tag{7}$$

where h is the plate thickness. The constitutive equations are

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \phi_{1,1} \\ \phi_{2,2} \\ 2\phi_{1,2} \end{bmatrix} \tag{8}$$

where $D_{ik}, i,k = 1,2,6,$ are the bending stiffness constants.
The general solution of the rotations and stress functions are given by [8]

$$\boldsymbol{\phi} = 2\mathrm{Re}\left[\mathbf{Af}\right],\ \boldsymbol{\psi} = 2\mathrm{Re}\left[\mathbf{Bf}\right] \tag{9}$$

where Re denotes the real part and

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$$\mathbf{A}=\left[\mathbf{a}_1 \quad \mathbf{a}_2\right],\mathbf{B}=\left[\mathbf{b}_1 \quad \mathbf{b}_2\right],\mathbf{f}(\mathbf{z})=\begin{bmatrix} f_1(z_1) \\ f_2(z_2)\end{bmatrix}, z_\alpha = x_1 + p_\alpha x_2. \tag{10}$$

Here $\mathbf{a}_\alpha$, $\mathbf{b}_\alpha$ and p_α are determined by

$$\begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}\begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{b}_\alpha \end{bmatrix} = p_\alpha \begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{b}_\alpha \end{bmatrix}, \tag{11}$$

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 1 \\ -\dfrac{D_{12}}{D_{22}} & -\dfrac{2D_{26}}{D_{22}} \end{bmatrix}, \mathbf{N}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{D_{22}} \end{bmatrix}, \mathbf{N}_3 = -\begin{bmatrix} D_{11}-\dfrac{D_{12}^2}{D_{22}} & 2\left(D_{16}-\dfrac{D_{12}D_{26}}{D_{22}}\right) \\ 2\left(D_{16}-\dfrac{D_{12}D_{26}}{D_{22}}\right) & 4\left(D_{66}-\dfrac{D_{26}^2}{D_{22}}\right) \end{bmatrix} \tag{12}$$

3. Stress Intensity Factors and Energy Release rate

Consider a semi-infinite crack with the crack tip located at the origin and the crack faces described by $\theta = \pm\pi$ as shown in Fig. 1.

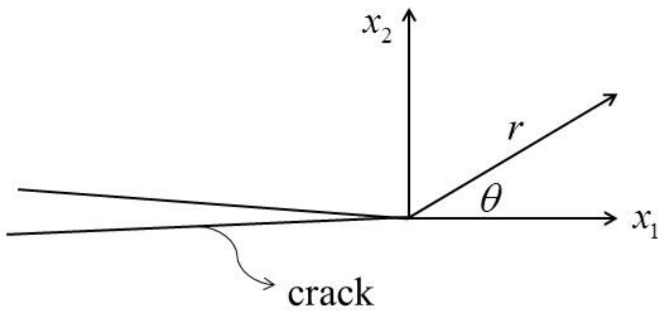

Figure 1. Configuration of a semi-infinite crack.

The traction-free conditions of the crack faces require that at $\theta = \pm\pi$,

$$\frac{\partial M_{r\theta}}{\partial r} + Q_\theta = 0, M_{\theta\theta} = 0. \tag{13}$$

The stress functions that satisfy Eq. (13) and give singular moments with bounded deflection and rotations are given by

$$\boldsymbol{\psi} = \sqrt{\frac{2r}{\pi}}\mathrm{Re}\left[\mathbf{B}\begin{bmatrix} \sqrt{\zeta_1} & 0 \\ 0 & \sqrt{\zeta_2} \end{bmatrix}\mathbf{B}^{-1}\right]\mathbf{K}, \tag{14}$$

where $\zeta_\alpha = \cos\theta + p_a \sin\theta$ and $\mathbf{K} = \left[K_2, K_1\right]^T$ is the momemt intensity factor. The corresponding moments ahead of the crack at $\theta = 0$ are

$$M_{22} = \frac{K_1}{\sqrt{2\pi r}}, M_{12} = \frac{1}{2\sqrt{2\pi r}}\left[\left(1-G_{21}\right)K_2 - G_{22}K_1\right], \tag{15}$$

where

$$\mathbf{G} = \mathrm{Re}\left[\mathbf{B}\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}\mathbf{B}^{-1}\right], \tag{16}$$

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From Eqs. (6) and (7) the stresses corresponding to Eq. (15) may be expressed as

$$\sigma_{22}=\frac{2x_3}{h}\frac{k_1}{\sqrt{2\pi r}},\ \sigma_{12}=\frac{1}{\sqrt{2\pi r}}\frac{x_3}{h}\left[\left(1-G_{21}\right)k_2-G_{22}k_1\right],$$
$$\sigma_{32}=-\frac{1}{\sqrt{2\pi r^3}}\frac{h}{16}\left[1-\left(\frac{2x_3}{h}\right)^2\right]\left[\left(1+G_{21}\right)k_2+G_{22}k_1\right], \tag{17}$$

where k_1 and k_2 are the bending stress intensity factors defined as

$$\mathbf{k}=\begin{bmatrix}k_2\\k_1\end{bmatrix}=\frac{6}{h^2}\mathbf{K}. \tag{18}$$

For orthotropic materials, σ_{12} in Eq. (17) reduces to

$$\sigma_{12}=\frac{1}{\sqrt{2\pi r}}\frac{2x_3}{h}\frac{1+v^*}{2+\rho+v^*}k_2,$$
$$\sigma_{32}=-\frac{1}{\sqrt{2\pi r^3}}\frac{h}{8}\left(1-\left(\frac{2x_3}{h}\right)^2\right)\frac{1+\rho}{2+\rho+v^*}k_2, \tag{19}$$

where $\rho=\sqrt{E_1E_2}/\left(2G_{12}\right)-v^*$, $v^*=\sqrt{v_1v_2}$, E_α, v_α and G_{12} are, respectively, Young's moduli, Poisson's ratios and the shear modulus. For isotropic materials $\rho=1$ and $v^*=v$, v being Poisson's ratio. The requirement of positive strain energy implies that $v^{*2}<1$. Values of typical composites are in the ranges $0<\rho<5$ and $0<v^*<1$. It is interesting to note that ρ and v^* of an orthotropic material are the same as those of the identical material but with a 90° rotation about the x_3 axis.
In contrast the stress intensity factors for the Reissner theory k_i' are defined by [9]

$$\sigma_{\alpha 2}=\frac{2x_3}{h}\frac{k_{3-\alpha}'}{\sqrt{2\pi r}},\ \sigma_{32}=\left(1-\left(\frac{2x_3}{h}\right)^2\right)\frac{k_3'}{\sqrt{2\pi r}}. \tag{20}$$

Equation (17) for σ_{22} is in the same form as Eq. (20). However, Eq. (17) shows that σ_{12} and σ_{32} are related to k_1 as well as k_2 in general and that the relationships contain material constants. This is quite different from the corresponding relationships in Eq. (20). Moreover, the transverse shear stress σ_{32} in (17) has a stronger $r^{-3/2}$ singularity.
The rotations corresponding to Eq. (14) are given by

$$\boldsymbol{\phi}=\sqrt{\frac{2r}{\pi}}\mathrm{Re}\left[\mathbf{A}\begin{bmatrix}\sqrt{\zeta_1}&0\\0&\sqrt{\zeta_2}\end{bmatrix}\mathbf{B}^{-1}\right]\mathbf{K}. \tag{21}$$

For orthotropic materials

$$u_\alpha=\pm\frac{2x_3}{h}\sqrt{\frac{2r}{\pi}}\frac{j^{1/2}}{E_1}\sqrt{2\left(1+\rho\right)}\frac{\sqrt{\left(1+v^*\right)\left(\rho+v^*\right)}}{\rho+2+v^*}k_{3-\alpha},$$
$$w=\mp\sqrt{\frac{2}{\pi}}\frac{4r^{3/2}}{3}\frac{j^{1/2}}{E_1h}\sqrt{2\left(1+\rho\right)}\frac{\sqrt{\left(1+v^*\right)\left(\rho+v^*\right)}}{\rho+2+v^*}k_2, \tag{22}$$

where $j=\sqrt{E_1/E_2}$.
The energy release rate *G* can be derived from the path-independant *J*-integral defined as

$$G=J=\int_{-h/2}^{h/2}\int_C\left(\frac{1}{2}\sigma_{ij}u_{i,j}n_1-\sigma_{ij}n_ju_{i,1}\right)dsdx_3, \tag{23}$$

where *C* is a contour in the mid-plane of the plate enclosing the crack tip. The result is

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$$G = \frac{h}{6}\mathbf{k}^T\hat{\mathbf{L}}^{-1}\mathbf{k}, \tag{24}$$

where

$$\hat{\mathbf{L}}^{-1} = -\frac{h^3}{12}\mathrm{Im}\left[\mathbf{AB}^{-1}\right] \tag{25}$$

and Im denotes the imaginary part. Eq. (24) becomes

$$G = \frac{h}{6}\frac{1}{E_1} j^{3/2}\sqrt{2(1+\rho)}\frac{\sqrt{(1+v^*)(\rho+v^*)}}{\rho+2+v^*}\left(k_1^2+\frac{k_2^2}{j}\right), \tag{26}$$

for orthotropic materials. When specialized to isotropic materials, the results derived here have been checked to agree with those in [6].
In comparison the energy release rate according to the Reissner theory is [9]

$$G = \frac{h}{6}\mathbf{k}'^T\mathbf{L}'^{-1}\mathbf{k}' + \frac{4h}{15}\frac{k_3'^2}{\mu}, \tag{27}$$

where $\mathbf{L}'$ is a 2×2 matrix and μ is a positive constant. Eq. (24) is in the same form as Eq. (27) except for the term involving k_3'. For orthotropic plates Eq. (27)
becomes [9]

$$G = \frac{h}{6}\frac{1}{E_1} j^{3/2}\sqrt{2(1+\rho)}\left[k_1'^2+\frac{k_2'^2}{j}\right]+\frac{4h}{15}\frac{k_3'^2}{\mu}. \tag{28}$$

4. Analytic solutions

Consider an infinite plate subjected to uniform bending $M_{22} = M_0$ or uniform twisting $M_{12} = H_0$ at infinity as shown in Fig. 2.

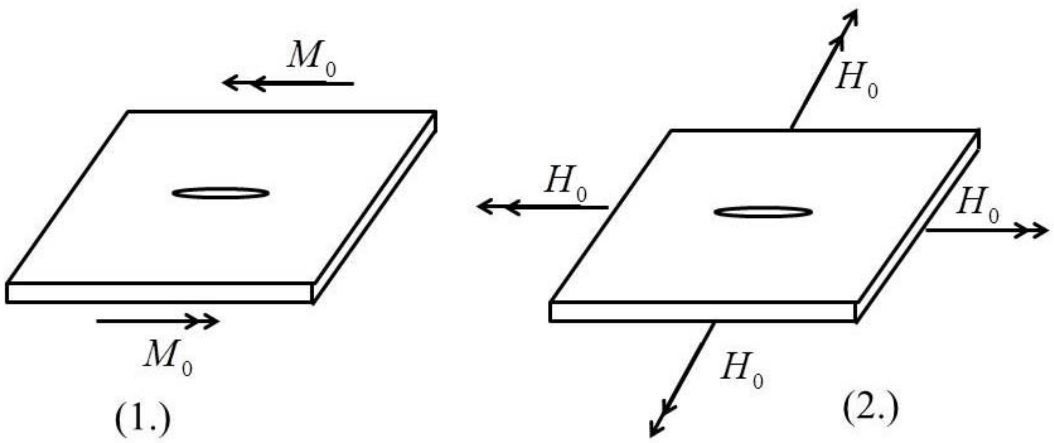

Figure 2. (1.) uniform bending $M_{22} = M_0$; (2.) uniform twisting $M_{12} = H_0$.

The stress intensity factors for the uniform bending case are derived as

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{6\sqrt{\pi a}M_0}{h^2}\begin{bmatrix} 1 \\ -\left(\hat{\mathbf{L}}^{-1}\right)_{12}/\left(\hat{\mathbf{L}}^{-1}\right)_{11} \end{bmatrix}, \tag{29}$$

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and for the uniform twisting case as

$$\mathbf{k} = \mathbf{0}. \tag{30}$$

A peculiar feature for isotropic plates was found by Zehnder and Hui [3] that if only twisting moments are applied, the stress intensity factors are zero and the stresses at the crack tip are finite. It is shown here in Eq. (30) that this is true even for general anisotropic plates.
The analytic results shown above may be used to derive the stress intensity factors for an inclined crack by the principle of superposition. As an example, consider a crack oriented at angle β to the x_1 axis and subjected to $M_{22} = M_0$ at infinity. The stress intensity factors can beobtained from those Eqs. (29) and (30) as

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = k_0 \cos^2\beta \begin{bmatrix} 1 \\ \dfrac{\cos\beta\sin\beta(1-j)}{\cos^2\beta + j\sin^2\beta} \end{bmatrix} \tag{31}$$

where $k_0 = 6\sqrt{\pi a} M_0 / h^2$. The corresponding energy release rate is

$$G = G_0 \frac{\sqrt{2(1+\rho)(1+v^*)(\rho+v^*)}}{2(2+\rho+v^*)} \frac{j^{3/2}\cos^4\beta}{\cos^2\beta + j\sin^2\beta} \tag{32}$$

where $G_0 = hk_0^2 / (3E_1)$. Equations (31) and (32) were used to calculate the stress intensity factors and energy release rate for a graphite-epoxy composite laminate with the following elastic constants [10]

$$E_1 = 153\text{GPa},\ E_2\text{=40.4, GPa},\ G_{12}\text{=29.3GPa},\ v_1\text{=1.83}, \tag{33}$$

or $\rho = 0.4$, $v^* = 0.942$, $j = 1.95$. The results of the stress intensity factors and energy release rate as a function of crack orientation are shown in Fig. 3. As the crack orientation increases from 0 to 90°, k_1 / k_0 and G / G_0, respectively, decrease monotonically from 1 and 1.37 to zero whereas k_2 / k_0 first decreases from zero, reaches a minimum value -0.255 at 26° and then increases to zero.

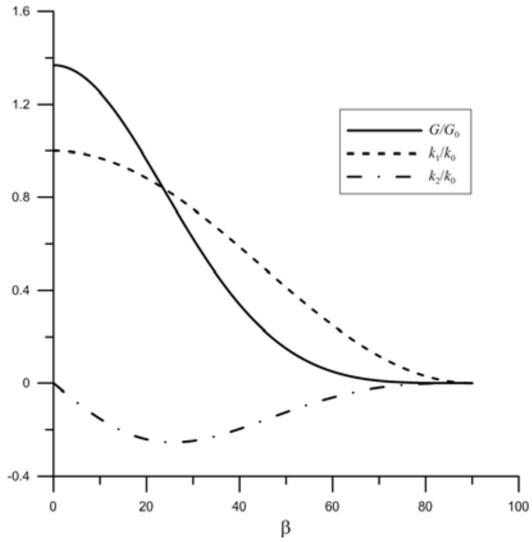

Figure 3. The stress intensity factors k_1, k_2 and energy release rate G for a crack inclined at angle β to the x_1 axis and subjected to $M_{22} = M_0$ at infinity.

Using the *J*-integral, Hui and Zender [6] showed that as $h \to 0$, the energy release rates based on either the classical plate theory or the Reissner theory are the same. Although the material they considered was isotropic, it is expected the equality still holds for orthotropic materials. For

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orthotropic plates under symmetric bending only $k_1^{'}$ and k_1 are nonzero.Thus equating Eqs. (26) and (28) leads to

$$k_1^{'} = \left(\frac{\sqrt{\left(1+v^*\right)\left(\rho+v^*\right)}}{\rho+2+v^*} \right)^{1/2} k_1 \tag{34}$$

With $\rho = 1$, $v^* = v$, Eq. (34) reduces to the result for isotropic material [6]. For uniform bending case, $k_1 = k_0$ and the Reissner theory stress intensity factor $k_1^{'} / k_0$ is plotted as a function of ρ for $v^* = 0, 0.25, 0.5, 0.75, 1$ in Fig.4. For a fixed value of v^*, $k_1^{'}$ first increases with ρ, reaches the maximum value $\sqrt[4]{\left(1+v^*\right)/8}$ at $\rho = 2 - v^*$, and then gradually decreases. For a fixed value of ρ, $k_1^{'}$ increases with v^*. It follows that the upper bound of $k_1^{'}$ is $k_0 / \sqrt{2} \approx 0.707 k_0$. The validity of Eq. (34) was checked with the numerical results in [10] for the graphite-epoxy composite laminate with the elastic constants of Eq. (33)and for the identical material but with the 90° rotation about the x_3 axis. The corresponding values $\rho = 0.4$ and $v^* = 0.942$ are the same in either case. From Eq. (34) $k_1^{'} = 0.695 k_0$ and this is very close to their numerical values as $h / a \to 0$.

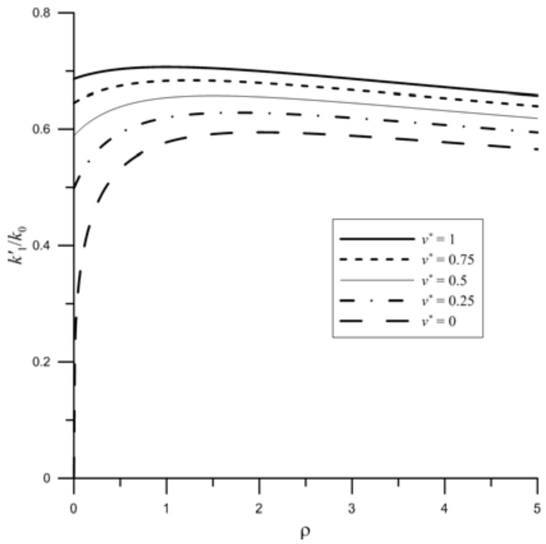

Figure 3. The Reissner theory stress intensity factor $k_1^{'}$ as a function of ρ and v^* for an orthotropic plate containing a crack parallel to of the symmetry axes and subjected to $M_{22} = M_0$ at infinity.

5. Conclusions

The stress and displacement fields near the tip of a through crack in a general anisotropic elastic plate are derived based on the classical plate theory. In particular, the stresses at the prolongation of the crack and the relative crack face displacements are derived. The expressions are more explicit and much simpler than those given by Sih and Chen [7].The results are used to define the stress intensity factors so that the facture mechanics parameters can be extracted numerically from the stresses or displacements in the vicinity of the crack tip. An explicit expression for the associated energy release rate in terms of the stress intensity factors is also obtained.

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