STRESSES ALONG THE HOLE BOUNDARY OF UNSYMMETRIC COMPOSITE LAMINATES VIA BOUNDARY ELEMENT METHOD

Chyanbin Hwu [1] and H.W. Chang [1]

[1] Institute of Aeronautics and Astronautics, National Cheng Kung University, Tainan, Taiwan, R.O.C.
Email: CHwu@mail.ncku.edu.tw, Web Page: http://www.iaa.ncku.edu.tw/~chwu

Keywords: hole, composite laminate, boundary element method, coupled stretching-bending analysis

Abstract

In boundary element analysis if the internal point coincides with the boundary nodes, which is usually required for the problem with holes, the associated integrals may become singular. Although this problem has been well treated for two-dimensional or plate bending analysis, no report can be found for the coupled stretching-bending analysis of unsymmetric composite laminates. To avoid the singular problem, in this paper the components of stress resultants, bending moments, in-plane strains and plate curvatures in the tangential direction are suggested to be calculated by finite difference method, whereas the components in normal direction are calculated with the aid of constitutive laws. To prove the correctness of the proposed calculation method, a simply supported unsymmetric laminate with two rectangular holes is analyzed and compared with the results obtained by finite element method.

1. Introduction

Due to the coupling of stretching and bending deformations, the stress analysis of the unsymmetric laminated plates becomes much more complicate than that of the metallic plates or the symmetric laminated plates. While the latter can be treated by considering only in-plane or plate bending analysis, the former can not avoid the coupling analysis. Through the use of Stroh-like complex variable formalism [1], the analytical closed-form solution of holes in laminates subjected to uniform tension/bending has been obtained in [2]. To make the analytical solution more useful for practical engineering analysis, the Green's function for infinite laminates with or without holes has also been obtained in our previous study [3, 4]. With these Green's functions, to effectively treat the coupled stretching-bending deformation of composite laminates, a boundary element was developed [5, 6]. Like the conventional boundary elements for two-dimensional or three-dimensional analysis, the boundary integral equation for the coupled stretching-bending analysis also contains singular integrals whose integrands may become infinity when the field point approaches to the source point [7]. To get an accurate result for the singular integral, several different methods have been proposed in our previous study [8, 9]. And hence, the displacements, slopes, in-plane tractions, transverse shear forces, and bending moments on the boundary nodes were all successfully obtained. Considering the source point of the boundary integral equation to be the internal point and taking derivatives with respect to the source point, the mid-plane strains, plate curvatures, stress resultants and bending moments at the internal points have also been obtained without involving the trouble of singular integrals if the internal point is not too close to the boundary nodes.

Due to the stress concentration around the hole boundary, it is always interested to know the stresses along the hole boundary. To solve this problem via boundary element method, suitable meshes should be made along the hole boundary, and hence one may encounter the trouble of finding the internal stresses of the boundary points because now the internal point coincides with the boundary nodes and

Excerpt from ISBN 978-3-00-053387-7

the singular integrals may occur again [10-12]. Since the displacements and slopes have been obtained on the boundary nodes, parts of strains and curvatures can be calculated approximately by using finite difference method [13]. In-plane tractions and bending moment in normal direction have also been obtained on the boundary nodes, parts of stress resultants and bending moments can then be calculated by using the transformation relation [14]. Through the use of the constitutive laws for the unsymmetric laminates, a computational method for the complete components of stress resultants, bending moments, in-plane strains and plate curvatures at the boundary nodes is proposed in this paper, which avoids the singular integrals. To verify the correctness of the proposed method, several numerical examples have been done and compared with the solutions calculated by the other methods.

2. Coupled Streching-Bending Theory of Composite Laminates

Consider a composite laminate with extensional, coupling and bending stiffness: A_{ij}, B_{ij}, D_{ij}. If the coupling stiffness B_{ij} is not equal to zero, which generally occurs for the unsymmetric laminates, the physical responses such as displacements, stresses and strains in plane direction and thickness direction will all couple each other. In Cartesian coordinate x_1-x_2-x_3, the governing equations satisfying the strain-displacement relations, the constitutive laws and the equilibrium equations for the coupled stretching-bending analysis of laminated plates can be written in terms of three unknown displacement functions u, v and w as

$$\begin{aligned}&A_{11}\frac{\partial^2 u}{\partial x_1^2}+2A_{16}\frac{\partial^2 u}{\partial x_1\partial x_2}+A_{66}\frac{\partial^2 u}{\partial x_2^2}+A_{16}\frac{\partial^2 v}{\partial x_1^2}+(A_{12}+A_{66})\frac{\partial^2 v}{\partial x_1\partial x_2}+A_{26}\frac{\partial^2 v}{\partial x_2^2}\\&-B_{11}\frac{\partial^3 w}{\partial x_1^3}-3B_{16}\frac{\partial^3 w}{\partial x_1^2\partial x_2}-(B_{12}+2B_{66})\frac{\partial^3 w}{\partial x_1\partial^2 x_2}-B_{26}\frac{\partial^3 w}{\partial x_2^3}=0,\end{aligned}\tag{1a}$$

$$\begin{aligned}&A_{16}\frac{\partial^2 u}{\partial x_1^2}+(A_{12}+A_{66})\frac{\partial^2 u}{\partial x_1\partial x_2}+A_{26}\frac{\partial^2 u}{\partial x_2^2}+A_{66}\frac{\partial^2 v}{\partial x_1^2}+2A_{26}\frac{\partial^2 v}{\partial x_1\partial x_2}+A_{22}\frac{\partial^2 v}{\partial x_2^2}\\&-B_{16}\frac{\partial^3 w}{\partial x^3}-(B_{12}+2B_{66})\frac{\partial^3 w}{\partial x_1^2\partial x_2}-3B_{26}\frac{\partial^3 w}{\partial x_1\partial x_2^2}-B_{22}\frac{\partial^3 w}{\partial x_2^3}=0,\end{aligned}\tag{1b}$$

$$\begin{aligned}&D_{11}\frac{\partial^4 w}{\partial x_1^4}+4D_{16}\frac{\partial^4 w}{\partial x_1^3\partial x_2}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+4D_{26}\frac{\partial^4 w}{\partial x_1\partial x_2^3}+D_{22}\frac{\partial^4 w}{\partial x_2^4}\\&-B_{11}\frac{\partial^3 u}{\partial x_1^3}-3B_{16}\frac{\partial^3 u}{\partial x_1^2\partial x_2}-(B_{12}+2B_{66})\frac{\partial^3 u}{\partial x_1\partial x_2^2}-B_{26}\frac{\partial^3 u}{\partial x_2^3}\\&-B_{16}\frac{\partial^3 v}{\partial x_1^3}-(B_{12}+2B_{66})\frac{\partial^3 v}{\partial x_1^2\partial x_2}-3B_{26}\frac{\partial^3 v}{\partial x_1\partial x_2^2}-B_{22}\frac{\partial^3 v}{\partial x_2^3}=q,\end{aligned}\tag{1c}$$

in which q is the lateral distributed load applied on the laminates. The general solution satisfying the governing equations (Eq. 1a,b,c) has been obtained and can be written in Stroh-like complex variable formalism [14]

$$\mathbf{u}_d=2\,\mathrm{Re}\{\mathbf{Af}(z)\},\qquad \boldsymbol{\phi}_d=2\,\mathrm{Re}\{\mathbf{Bf}(z)\},\tag{2}$$

in which Re stands for the real part of a complex number, $\mathbf{u}_d$ and $\boldsymbol{\phi}_d$ are the vectors of generalized displacements and stress functions, **A** and **B** are the Stroh's eigenvector matrices, and $\mathbf{f}(z)$ is a vector of complex function.

Consider an infinite laminate subjected to a concentrated force $\hat{\mathbf{f}}=(\hat{f}_1,\hat{f}_2,\hat{f}_3)$ and moment $\hat{\mathbf{m}}=(\hat{m}_1,\hat{m}_2,\hat{m}_3)$ at point $\hat{\mathbf{x}}=(\hat{x}_1,\hat{x}_2,0)$. The solution to this problem, which is usually called *Green's function*, can be written in the form of (Eq. 2) where the complex function vector $\mathbf{f}(z)$ is [3]

Excerpt from ISBN 978-3-00-053387-7

$$\mathbf{f}(z)=\frac{1}{2\pi i}\Big\{<\ln(z_\alpha-\hat{z}_\alpha)>\mathbf{A}^T\hat{\mathbf{p}}+\hat{f}_3<(z_\alpha-\hat{z}_\alpha)[\ln(z_\alpha-\hat{z}_\alpha)-1]>\mathbf{A}^T\mathbf{i}_3 \\ -\hat{m}_3<\frac{1}{z_\alpha-\hat{z}_\alpha}>\mathbf{A}^T\mathbf{i}_2\Big\}, \tag{3a}$$

and

$$\hat{\mathbf{p}}=(\hat{f}_1,\hat{f}_2,\hat{m}_2,-\hat{m}_1)^T,\ \mathbf{i}_1=(1,0,0,0)^T,\ \mathbf{i}_2=(0,1,0,0)^T,\ \mathbf{i}_3=(0,0,1,0)^T, \\ z_\alpha=x_1+\mu_\alpha x_2,\quad \hat{z}_\alpha=\hat{x}_1+\mu_\alpha\hat{x}_2,\quad \alpha=1,2,3,4. \tag{3b}$$

In the above, the angular bracket <> stands for the diagonal matrix in which each component is varied according to its subscript α, the superscript T denotes transpose of a matrix or vector, and μ_α is the Stroh's eigenvalue.

Substituting the Green's function (Eq. 3) into the boundary integral equation for the coupled stretching-bending analysis [5], and using suitable interpolation functions, the system of linear algebraic equations for boundary element method (BEM) can be written as [6]

$$\sum_{j=1}^{N}\left\{\mathbf{Y}_{ij}\mathbf{u}_j-\mathbf{G}_{ij}\mathbf{t}_j\right\}+\sum_{k=1}^{N_c^*}\left\{\mathbf{p}_{ik}^*w^{(k)}-\mathbf{w}_{ik}^*t_c^{(k)}\right\}=\mathbf{q}_i^*,\qquad i=1,2,\ldots,N+N_c, \tag{4}$$

in which $\mathbf{Y}_{ij}$ and $\mathbf{G}_{ij}$ are matrices of influence coefficients, $\mathbf{u}_j$ and $\mathbf{t}_j$ are vectors of nodal displacements and nodal tractions, $\mathbf{p}_{ik}^*$ and $\mathbf{w}_{ik}^*$ are vectors related to the fundamental solutions of corner forces and deflection, $w^{(k)}$ and $t_c^{(k)}$ are the deflection and corner force of the kth corner, $\mathbf{q}_i^*$ is a vector related to the lateral load, N is the number of regular node, and N_c^* is related to the number of corners N_c by

$$N_c^*=\begin{cases}N_c, & \text{if the source point is not a corner point;}\\ N_c-1, & \text{if the source point is a corner point.}\end{cases} \tag{5}$$

3. Solutions at the boundary nodes of BEM

Having the detailed expressions for each matrix and vector shown in (Eq. 4), the computer program of boundary element for the coupled stretching-bending analysis can be coded. With this program, the users should input the material properties and boundary conditions for the problems they concern. The displacements and tractions on the boundary nodes of (Eq. 4) can then be solved. In other word,

$$\mathbf{u}_j=\begin{Bmatrix}u\\ v\\ w\\ \beta_n\end{Bmatrix}_j,\ \mathbf{t}_j=\begin{Bmatrix}T_x\\ T_y\\ V_n\\ M_n\end{Bmatrix}_j,\ j=1,2,\ldots,N, \tag{6}$$

are known at the boundary nodes. In (Eq. 6), u, v and w are the displacements in the direction of x_1, x_2, and x_3, β_n is the negative slope of deflection in normal direction; T_x and T_y are the x_1 and x_2 components of surface traction; V_n and M_n are the effective transverse shear force and bending moment on the surface with normal direction.

With the result of (Eq. 6), the values of strains, curvatures, stress resultants and bending moments at the boundary nodes can be calculated by the following steps (see Fig. 1).

Step 1: calculate $\varepsilon_s,\kappa_s,\kappa_{sn}$ *through the nodal displacement.*

Excerpt from ISBN 978-3-00-053387-7

$$\begin{aligned}
\varepsilon_s &= \frac{\partial u_s}{\partial s} \cong \frac{(u^+ - u^-)\cos\theta + (v^+ - v^-)\sin\theta}{(x_1^+ - x_1^-)\cos\theta + (x_2^+ - x_2^-)\sin\theta}, \\
\kappa_s &= -\frac{\partial^2 w}{\partial s^2} = \frac{\partial^2 \beta_s}{\partial s} = \frac{(\beta_s^+ - \beta_s^-)}{(x_1^+ - x_1^-)\cos\theta + (x_2^+ - x_2^-)\sin\theta}, \\
\kappa_{sn} &= -2\frac{\partial^2 w}{\partial s \partial n} = 2\frac{\partial^2 \beta_n}{\partial s} = \frac{2(\beta_n^+ - \beta_n^-)}{(x_1^+ - x_1^-)\cos\theta + (x_2^+ - x_2^-)\sin\theta},
\end{aligned} \tag{7a}$$

where

$$\beta_s^+ = -\left.\frac{\partial w}{\partial s}\right|_{\xi^+}, \quad \beta_s^- = -\left.\frac{\partial w}{\partial s}\right|_{\xi^-}, \quad \beta_n^+ = -\left.\frac{\partial w}{\partial n}\right|_{\xi^+}, \quad \beta_n^- = -\left.\frac{\partial w}{\partial n}\right|_{\xi^-}. \tag{7b}$$

Note that $\beta_s^\pm$ and $\beta_n^\pm$ are different from the nodal values of β_s and β_n at points $\xi^\pm$ since the direction angles of these two points are θ^+ and θ^- instead of θ. One can calculate $\beta_s^\pm$ and $\beta_n^\pm$ from the nodal values of β_s and β_n at points $\xi^\pm$ through the coordinate transformation of vector.

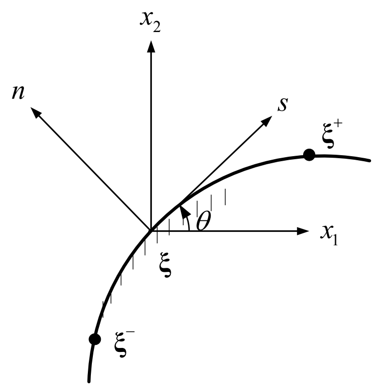

Figure 1. The local coordinates of $s-n$, and the direction angle θ.

Step 2: calculate N_n, N_{sn}, M_n through the nodal traction.

$$N_n = -T_x \sin\theta + T_y \cos\theta, \;\; N_{ns} = T_x \cos\theta + T_y \sin\theta, \;\; M_n = M_n. \tag{8}$$

Step 3: calculate $N_s, \varepsilon_n, \gamma_{sn}, M_s, \kappa_n, M_{sn}$ through the constitutive law.

$$\begin{bmatrix} 1 & -A'_{12} & -A'_{16} & 0 & -B'_{12} & 0 \\ 0 & -A'_{22} & -A'_{26} & 0 & -B'_{22} & 0 \\ 0 & -A'_{62} & -A'_{66} & 0 & -B'_{62} & 0 \\ 0 & -B'_{12} & -B'_{16} & 1 & -D'_{12} & 0 \\ 0 & -B'_{22} & -B'_{26} & 0 & -D'_{22} & 0 \\ 0 & -B'_{62} & -B'_{66} & 0 & -D'_{62} & 1 \end{bmatrix} \begin{Bmatrix} N_s \\ \varepsilon_n \\ \gamma_{sn} \\ M_s \\ \kappa_n \\ M_{sn} \end{Bmatrix} = \begin{bmatrix} A'_{11} & 0 & 0 & B'_{11} & 0 & B'_{16} \\ A'_{21} & -1 & 0 & B'_{21} & 0 & B'_{26} \\ A'_{61} & 0 & -1 & B'_{61} & 0 & B'_{66} \\ B'_{11} & 0 & 0 & D'_{11} & 0 & D'_{16} \\ B'_{21} & 0 & 0 & D'_{21} & -1 & D'_{26} \\ B'_{61} & 0 & 0 & D'_{61} & 0 & D'_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_s \\ N_n \\ N_{sn} \\ \kappa_s \\ M_n \\ \kappa_{sn} \end{Bmatrix}, \tag{9}$$

where $A'_{ij}, B'_{ij},$ and D'_{ij} are the extensional, coupling and bending stiffnesses at the s-n coordinate.

Step 4: calculate $N_x, N_y, N_{xy}, M_x, M_y, M_{xy}$ through transformation relation.

Excerpt from ISBN 978-3-00-053387-7

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \mathbf{T} \begin{Bmatrix} N_s \\ N_n \\ N_{sn} \end{Bmatrix}, \quad \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \mathbf{T} \begin{Bmatrix} M_s \\ M_n \\ M_{sn} \end{Bmatrix}, \tag{10a}$$

where

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}. \tag{10b}$$

Step 5: calculate $\varepsilon_x, \varepsilon_y, \gamma_{xy}, \kappa_x, \kappa_y, \kappa_{xy}$ *through transformation relation.*

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \mathbf{T} \begin{Bmatrix} \varepsilon_s \\ \varepsilon_n \\ \gamma_{sn} \end{Bmatrix}, \quad \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \mathbf{T} \begin{Bmatrix} \kappa_s \\ \kappa_n \\ \kappa_{sn} \end{Bmatrix}. \tag{11}$$

From the above steps, we see that the singular integrals which will generally be encountered in the calculation of stresses and strains of internal points are avoided technically.

4. Numerical Examples

Consider an unsymmetric composite laminate [0/45/90/30/-45/90/45/-60] composed of carbon/epoxy fiber-reinforced laminae. The thickness of each lamina is 1 mm, and the elastic properties of carbon /epoxy are

$$E_1 = 11.8 \text{ GPa}, \; E_2 = 5.9 \text{ GPa}, \; G_{12} = 0.69 \text{ GPa}, \; \nu_{12} = 0.071,$$

where the symbols E, G and ν denote, respectively, Young's modulus, shear modulus and Poisson's ratio, and the subscripts 1 and 2 denote, respectively, the fiber direction and the direction transverse to the fiber. The laminate contains two identical rectangular holes, and is simply supported on all edges of the outer boundary as well as the upper and lower edges of the holes (see Fig. 2). A uniformly lateral distributed load $q_0 = -6.5$ kPa is applied on the upper surface of the laminated plate.

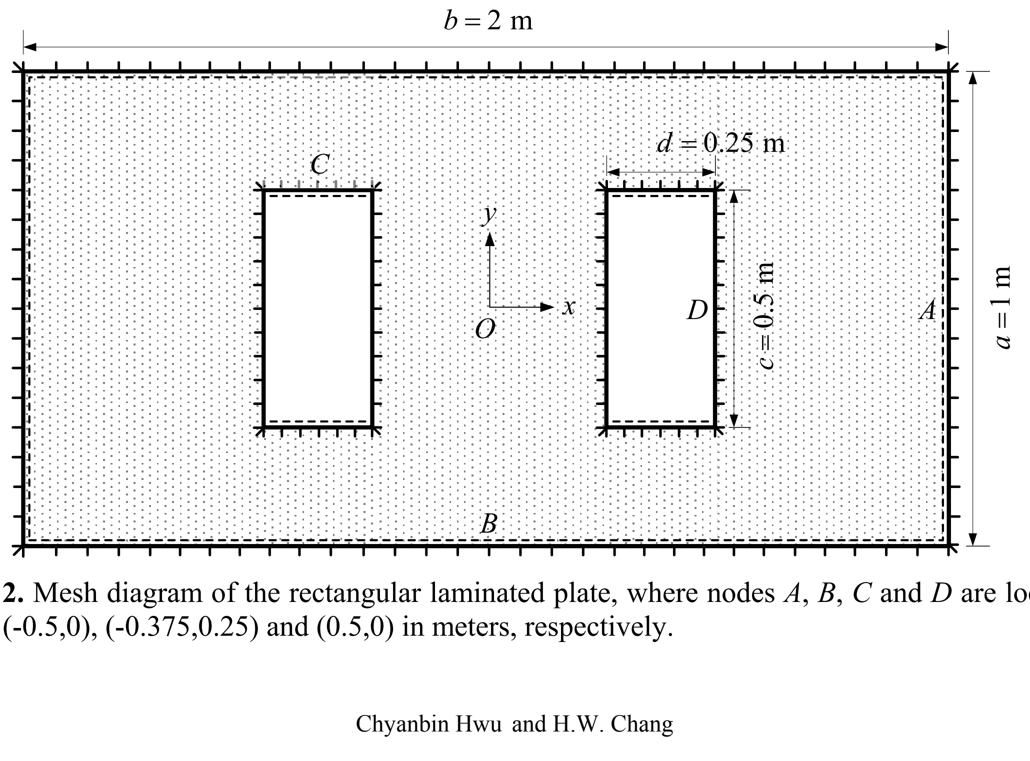

Figure 2. Mesh diagram of the rectangular laminated plate, where nodes A, B, C and D are located at (1.0,0), (-0.5,0), (-0.375,0.25) and (0.5,0) in meters, respectively.

Excerpt from ISBN 978-3-00-053387-7

After the convergence test, 156 elements were used for the meshes of plate boundaries. The normalized stress resultants at the four boundary nodes *A*, *B*, *C*, *D* and plate center *O* are shown in Table 1, and the contour plots of the deflections by the present BEM and ANSYS are also shown in Fig. 3(a) and Fig. 3(b), in which the deflection at the center of the plate are -6.020 mm and -6.197 mm, respectively. The normalized stress resultants is defined as

$$\begin{aligned} \bar{N}_x &= \frac{10N_x}{q_0a^2}, \quad \bar{N}_y = \frac{10N_y}{q_0a^2}, \quad \bar{N}_{xy} = \frac{10N_{xy}}{q_0a^2}, \\ \bar{M}_x &= \frac{10M_x}{q_0a^2}, \quad \bar{M}_y = \frac{10M_y}{q_0a^2}, \quad \bar{M}_{xy} = \frac{10M_{xy}}{q_0a^2}. \end{aligned} \tag{12}$$

As shown in Table 1 and Fig.3, the correctness of the proposed calculation method is now verified by the comparison with the solutions obtained from finite element software ANSYS.

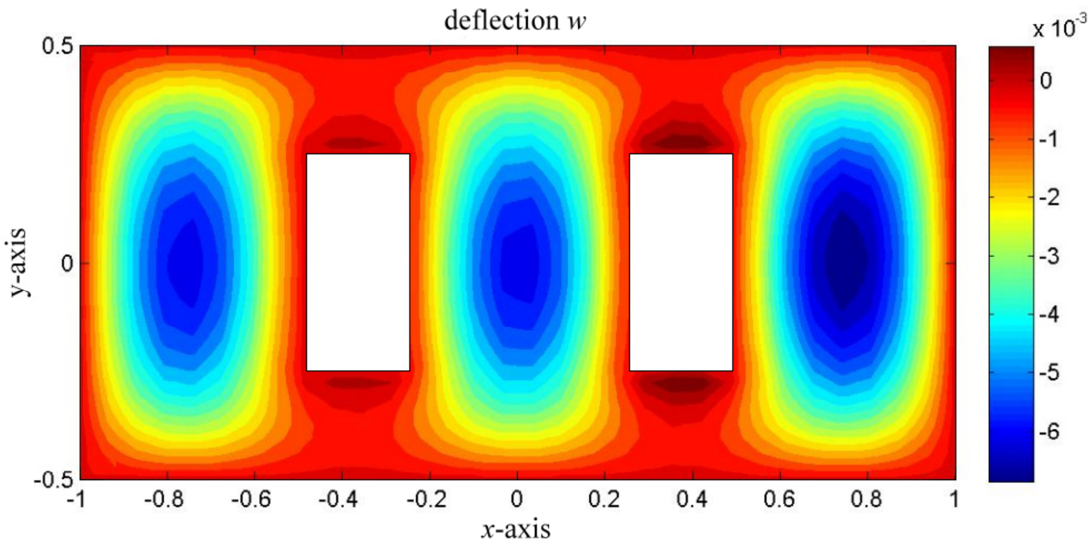

Figure 3(a). Deflection contour by the present BEM.

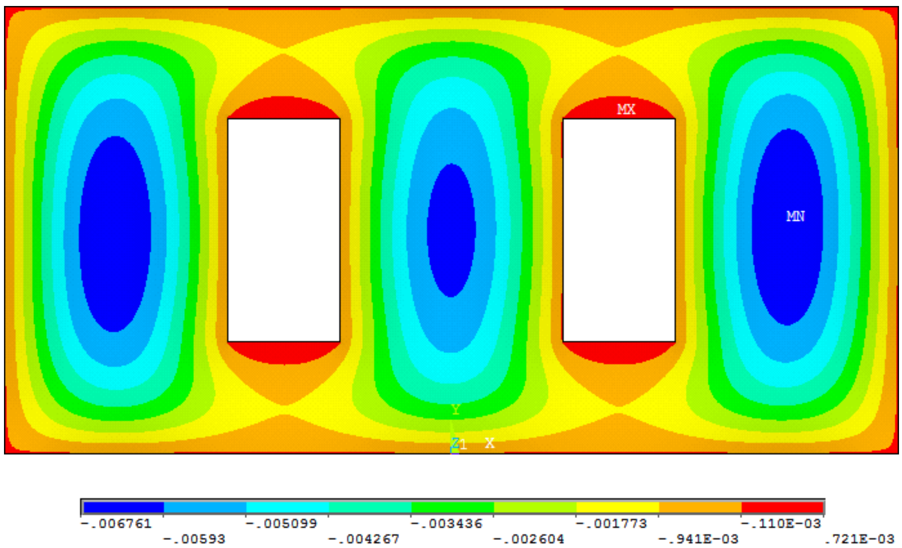

Figure 3(b). Deflection contour by ANSYS.

Excerpt from ISBN 978-3-00-053387-7

Table 1. The stress resultants at the selected points of composite laminate.

	$\bar{N}_x$	$\bar{N}_y$	$\bar{N}_{xy}$	$\bar{M}_x$	$\bar{M}_y$	$\bar{M}_{xy}$
Point *A*						
Present BEM	-9.6615	-3.2949	-2.70	0.0091	0.0022	0.0016
ANSYS	-10.189	-4.1287	-2.8739	0.	-0.0040	0.0030
Point *B*						
Present BEM	3.3709	11.503	-0.9430	0.0111	0.0071	-0.0004
ANSYS	3.7041	11.8536	-1.0473	0.0014	0.	-0.0046
Point *C*						
Present BEM	7.5066	0.0119	-0.0674	-0.0444	0.0010	-0.0016
ANSYS	7.4983	0.	0.	-0.0115	0.	0.0011
Point *D*						
Present BEM	-10.555	-3.6283	-2.9812	0.0089	0.0028	0.0006
ANSYS	-11.586	-4.3851	-3.5234	0.	-0.0038	-0.0002
Point *O* (Center of the plate)						
Present BEM	-9.9141	3.9524	-0.2275	0.2195	0.1109	0.0132
ANSYS	-10.106	4.2288	-0.2565	0.2273	0.1133	0.0136

5. Conclusions

Since no singular integral involved in the computational procedure stated in Section 3, the stresses along the hole boundary of unsymmetric composite laminates can be calculated directly from the displacements and tractions of the boundary nodes. The comparison with the results obtained by ANSYS for hole problems shows that the method proposed in this paper is accurate and efficient.

Acknowledgments

The authors would like to thank Ministry of Science and Technology, TAIWAN, R.O.C. for support through Grants MOST 104-2221-E-006-138-MY3.

References

[1] C. Hwu. Stroh-Like Formalism for the coupled stretching-bending Analysis of composite laminates. *International Journal of Solids and Structures,* 40(13-14):3681-3705, 2003.

[2] M.C. Hsieh and C. Hwu. Explicit Solutions for the Coupled Stretching-Bending Problems of Holes in Composite Laminates. *International Journal of Solids and Structures*, 40(15):3913-3933, 2003.

[3] C. Hwu. Green's function for the composite laminates with bending extension coupling. *Composite Structures*, 63:283-292, 2004.

[4] C. Hwu. Green's Functions for Holes/Cracks in Laminates with Stretching-Bending Coupling. *ASME Journal of Applied Mechanics*, 72:282-289, 2005.

[5] C. Hwu. Boundary integral equations for general laminated plates with coupled stretching-bending deformation. *Proceedings of the Royal Society, Series A*, 466:1027-1054, 2010.

[6] C. Hwu. Boundary element formulation for the coupled stretching-bending analysis of thin laminated plates. *Engineering Analysis with Boundary Elements*. 36(6):1027-1039, 2012.

[7] C.A. Brebbia, J.C.F. Telles, L.C. Wrobel. *Boundary Element Techniques: Theory and*

Excerpt from ISBN 978-3-00-053387-7

Applications in Engineering. Springer, 1984.

[8] C. Hwu and H.W. Chang. Coupled Stretching-Bending Analysis of Laminated Plates with Corners via Boundary Elements. *Composite Structures*. 120:300-314, 2015.

[9] C. Hwu and H.W. Chang. Singular Integrals in Boundary Elements for Coupled Stretching-Bending Analysis of Unsymmetric Laminates. *Composite Structures*.132:933-943, 2015.

[10] C.C. Chien, H. Rajiyah, S.N. Atluri. On the evaluation of Hyper-singular integrals arising in the boundary element method for linear elasticity. *Computational Mechanics*. 8:57-70, 1991.

[11] M. Guiggiani. Hypersingular formulation for boundary stress evaluation. *Engineering Analysis with Boundary Elements*. 13:169-179, 1994.

[12] E. Ehao and S. Lan. Boundary stress calculation – a comparison study. *Composite Structures*. 71:77-85, 1999.

[13] C. Hwu and H.W. Chang. Complete solutions at or near the boundary nodes of boundary elements for coupled stretching-bending analysis. *To be submitted for publication.*

[14] C. Hwu. *Anisotropic Elastic Plates*. Springer, New York, 2010.

Excerpt from ISBN 978-3-00-053387-7