ON LAY-UP OPTIMIZATION OF ANISOTROPIC COMPOSITE PLATES IN POST-BUCKLING

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Abstract

The composite plates in post-buckling are considered. The plates are made of a given number of layers. The plate lay-up is symmetric. The layer lay-up angles are varied (point-wise) to maximize structural stiffness. The measure of stiffness is the structural potential energy. The von Karman equation is used for post-buckling behavior. The boundary conditions are simple support or clamped. For every layer the first order necessary optimality conditions are obtained. The conditions for layer angles after combining contain a sum of two terms. The first one is related to bending, the second one is related to in-plane non-linear strain state. The first term is equal to zero when the twisting moment in principal curvature axes is equal to zero. The second term is equal to zero when the shear flow in principal strain axes is equal to zero. Co-axiality of two pairs of structural tensors is important for optimality. The plate-wise layer angle variation case and optimum conditions for it are indicated and analyzed also. The orthotropic orientation optimization problem is a particular case of the above-described problems. Illustrating examples demonstrating peculiarities of the optimal solutions are presented. The examples may be used as benchmarks for optimization software.

1. Introduction

Post-buckling of thin-walled composite plates and their optimization are very important for modern aerospace, marine and civil applications of composite materials. Starting from 90-ies numerous papers are devoted to lay-up optimization of the plates. The important, from our viewpoint, and the latest publications devoted to the post-buckling optimization are (the list is not exhaustive) [1]-[8]. The foundations of the post-buckling theory are described in [9]. Other interesting publications may be found in references of the above papers.

All published papers in the area are devoted to numerical optimization studies. In the majority of the papers no flexural/2D anisotropy is considered (see e.g. [10]). Various optimization criteria like end-shortening, maximum strain, maximum deflection, stiffness estimation, etc., are used.

Talking about papers considering some description of the post-buckling stiffness (the current study deals with the theoretical analysis of such a problem), the paper [6] should be indicated. In the paper the compressed orthotropic laminated plates in post-buckling are considered. The solution of the corresponding von Karman and strain compatibility equations is determined numerically using the Airy function and the deflection described by several trigonometric terms satisfying boundary conditions. The estimation of the post-buckling stiffness in every direction is calculated as the derivative of the corresponding load w.r.t. the proper compressive strain. Maximization of the initial post-buckling stiffness (with given compressive load ratio for x and y directions) is performed numerically.

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No influence of flexural/2D anisotropy on lay-up in post-buckling is theoretically considered in available publications. The numerical experience till now is rather limited. The main target of the current paper is to reveal regularities for optimal lay-up of composite plates in post-buckling. The optimality conditions to be derived are necessary for validation and verification of numerical solutions.

2. Theoretical consideration

Anisotropic thin composite plate of constant thickness is considered. The Kirchhoff assumptions are used. The plate is of symmetric lay-up with 2*N* layers. Clamped or simple support boundary conditions are considered. The contour is assumed to be a piece-wise smooth one. Non-linear terms (w.r.t. deflections) in the Green strain tensor are taken into account. The plate is located in (x, y) plane.

The Von Karman equation, together with the strain compatibility condition, is used for plate post-buckling behavior description [10], [11].

The best lay-up search is based here on total structural potential energy stationarity principle (for isotropic plates see the derivation of the principle in [11], Section 8). The variational principle is a kinematic one.

Let us present the kinematic variational principle for composite plates in the above assumptions. The quantities (u, v, w) are the plate displacements in the Cartesian coordinates (x, y, z), w is the deflection. The corresponding components of the Green strain tensor in the plate mid-plane with account of non-linear terms are:

$$e_{xx0} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 ;\ e_{yy0} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 ;\ e_{xy0} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \tag{1}$$

The components of the tensor of mid-plane curvatures:

$$k_x = -\frac{\partial^2 w}{\partial x^2} ;\ k_y = -\frac{\partial^2 w}{\partial y^2} ;\ k_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \tag{2}$$

The following vectors are introduced (below we follow the engineering approach of [12]:

$$\overline{\varepsilon^0}^T = \left(e_{xx0}, e_{yy0}, 2e_{xy0}\right) ;\ \overline{k}^T = \left(k_x, k_y, 2k_{xy}\right) \tag{3}$$

where *T* means transposing.

The strain potential energy in case of large deflections (see [11], Section 8) is:

$$\Pi = \int dS\left(\frac{1}{2}\overline{\varepsilon^0}^T A\overline{\varepsilon^0} + \frac{1}{2}\overline{k}^T D\overline{k}\right) \tag{4}$$

where *A* and *D* are the (3*3) matrices describing 2D-mid-plane behavior and bending of the plate, respectively [12].

The structural potential energy:

$$U = \int dS\left(\frac{1}{2}\overline{\varepsilon^0}^T A\overline{\varepsilon^0} + \frac{1}{2}\overline{k}^T D\overline{k}\right) - W \tag{5}$$

where *W* is the potential of external forces depending on the forces, on the displacements at loaded part of the external boundary and, if any, on external pressure and deflections within the plate. The kinematic variational principle states that in equilibrium the structure satisfies

$$\delta U = 0 \tag{6}$$

for all kinematically admissible (u, v, w) displacements.

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The principle (6) leads to the equilibrium equation [10]

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}- \\ -N_x\frac{\partial^2 w}{\partial x^2}-N_y\frac{\partial^2 w}{\partial y^2}-2N_{xy}\frac{\partial^2 w}{\partial x\partial y}=0 \tag{7}$$

where N_x, N_y, N_{xy} are the stress flows (see [12] for the usual sign convention). The plate deflections and the mid-plane strains must also satisfy the compatibility equation:

$$\frac{\partial^2\varepsilon_{xx0}}{\partial y^2}+\frac{\partial^2\varepsilon_{yy0}}{\partial x^2}-2\frac{\partial^2\varepsilon_{xy0}}{\partial x\partial y}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{8}$$

The orthotropic lamina is shown at the Fig. 1 (see [12], Section 2.6). The stress tensor components within the lamina in coordinates (*x*, *y*) are:

$$\begin{pmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{pmatrix}=\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ 2\varepsilon_{xy}\end{pmatrix} \tag{9}$$

where $\overline{Q}$ is the lamina stiffness matrix and $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ are the lamina strain tensor components.

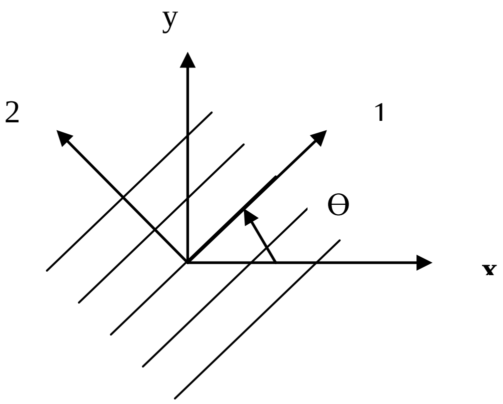

Fig. 1. Orthotropic lamina with material orientation (1, 2).

The components of the lamina stiffness matrix are (see [12], Section 2.6):

$$\begin{aligned}\overline{Q}_{11}&=U_1+U_2\cos 2\theta+U_3\cos 4\theta\\ \overline{Q}_{12}&=U_4-U_3\cos 4\theta\\ \overline{Q}_{22}&=U_1-U_2\cos 2\theta+U_3\cos 4\theta\\ \overline{Q}_{16}&=\frac{1}{2}U_1\sin 2\theta+U_3\sin 4\theta\\ \overline{Q}_{26}&=\frac{1}{2}U_1\sin 2\theta-U_3\sin 4\theta\\ \overline{Q}_{66}&=\frac{1}{2}(U_1-U_4)-U_3\cos 4\theta\end{aligned} \tag{10}$$

where U_1, U_2, U_3, U_4 are the elastic constants of the orthotropic layer.

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The derivatives of the lamina stiffness matrix $\overline{Q}_{ij}$ w.r.t. the angle θ are, according to (10):

$$\begin{aligned} \frac{d\overline{Q}_{11}}{d\theta} &= -2U_2 \sin 2\theta - 4U_3 \sin 4\theta = -4\overline{Q}_{16} \\ \frac{d\overline{Q}_{22}}{d\theta} &= 2U_2 \sin 2\theta - 4U_3 \sin 4\theta = 4\overline{Q}_{26} \\ \frac{d\overline{Q}_{12}}{d\theta} &= 4U_3 \sin 4\theta = 2(\overline{Q}_{16} - \overline{Q}_{26}) \end{aligned} \tag{11}$$

Determination of A_{ij} and D_{ij} via lamina parameters is made according to the general formulas [12], *ij=11, 12, 22, 16, 26, 66*:

$$A_{ij} = \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right) \tag{12}$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right) \tag{13}$$

where z_k, z_{k-1} are the top and bottom *z*-coordinates of the layer.

The derivatives of the stiffness quantities w.r.t. lamina angles θ_k are, *ij=11, 12, 22; k=1,…,N*:

$$\begin{aligned} \frac{dA_{ij}}{d\theta_k} &= (z_k - z_{k-1})\frac{d\left(\overline{Q}_{ij}\right)_k}{d\theta_k} \\ \frac{dD_{ij}}{d\theta_k} &= \frac{1}{3}(z_k^3 - z_{k-1}^3)\frac{d\left(\overline{Q}_{ij}\right)_k}{d\theta_k} \end{aligned} \tag{14}$$

where θ_k is the layer orientation angle.

Below we consider the structural potential energy *U* as a measure of structural stiffness. The goal of lay-up optimization is to maximize the stiffness in post-buckling. We assume that the lamina orientation angles θ_k are smooth functions of the mid-plane location (*x*,*y*).

Following usual variational procedure and taking into account (6), we obtain the first order necessary optimality conditions in case of point-wise variations of the layer orientation angles:

$$\left.\frac{\partial \Pi}{\partial \theta_k}\right|_{u,v,w=const} = 0, \quad k=1,\dots,N \tag{15}$$

Further we will calculate the first term in (4), (15), corresponding to mid-plane 2D deformations, in the axes of principal 2D mid-plane strains $\varepsilon_1, \varepsilon_2$, where $\varepsilon_1 \ge \varepsilon_2$. The second term in (4), (15), corresponding to plate bending, we will calculate in the axes of principal plate curvatures k_1, k_2, where $k_1 \ge k_2$. Making necessary transformations in (15), we obtain:

$$\frac{1}{2}\left\{\left(\varepsilon_1, \varepsilon_2\right)\begin{bmatrix} \frac{dA_{11}^{pr.str.}}{d\theta_k} & \frac{dA_{12}^{pr.str.}}{d\theta_k} \\ \frac{dA_{12}^{pr.str.}}{d\theta_k} & \frac{dA_{22}^{pr.str.}}{d\theta_k} \end{bmatrix}\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}\right\} + \frac{1}{2}\left\{\left(k_1, k_2\right)\begin{bmatrix} \frac{dD_{11}^{pr.cur.}}{d\theta_k} & \frac{dD_{12}^{pr.cur.}}{d\theta_k} \\ \frac{dD_{12}^{pr.cur.}}{d\theta_k} & \frac{dD_{22}^{pr.cur.}}{d\theta_k} \end{bmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\right\} = 0 \tag{16}$$

where *pr.str.* and *pr.cur.* correspond to principal mid-plane strain axes and principal curvature axes, respectively.

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Let us consider the first term in the relation (16). Substituting to (16) the relations (11)-(14), we obtain for every k-th layer, k=1,…,N:

$$\begin{Bmatrix} \varepsilon_1^2\left[-2U_2\sin 2(\theta_k-\varphi)-4U_3\sin 4(\theta_k-\varphi)\right]+ \\ +\varepsilon_2^2\left[2U_2\sin 2(\theta_k-\varphi)-4U_3\sin 4(\theta_k-\varphi)\right]+2\varepsilon_1\varepsilon_2 4U_3\sin 4(\theta_k-\varphi) \end{Bmatrix}\frac{1}{2}\left(z_k-z_{k-1}\right) \qquad (17)$$

where φ is the angle between the global x axis and the ε_1 axis.

Analogously the second term gives for every k-th layer, k=1,…,N:

$$\begin{Bmatrix} k_1^2\left[-2U_2\sin 2(\theta_i-\psi)-4U_3\sin 4(\theta_i-\psi)\right]+ \\ +k_2^2\left[2U_2\sin 2(\theta_i-\psi)-4U_3\sin 4(\theta_i-\psi)\right]+2k_1k_2 4U_3\sin 4(\theta_i-\psi) \end{Bmatrix}\frac{1}{6}\left(z_k^3-z_{k-1}^3\right) \qquad (18)$$

where ψ is the angle between the global x axis and the k_1 axis.

The sum of (17) and (18) equated to zero is the first order necessary optimality condition at every layer, k=1,...,N

$$\begin{Bmatrix} \varepsilon_1^2\left[-2U_2\sin 2(\theta_k-\varphi)-4U_3\sin 4(\theta_k-\varphi)\right]+ \\ +\varepsilon_2^2\left[2U_2\sin 2(\theta_k-\varphi)-4U_3\sin 4(\theta_k-\varphi)\right]+2\varepsilon_1\varepsilon_2 4U_3\sin 4(\theta_k-\varphi) \end{Bmatrix}\frac{1}{2}\left(z_k-z_{k-1}\right)+$$
$$+\begin{Bmatrix} k_1^2\left[-2U_2\sin 2(\theta_i-\psi)-4U_3\sin 4(\theta_i-\psi)\right]+ \\ +k_2^2\left[2U_2\sin 2(\theta_i-\psi)-4U_3\sin 4(\theta_i-\psi)\right]+2k_1k_2 4U_3\sin 4(\theta_i-\psi) \end{Bmatrix}\frac{1}{6}\left(z_k^3-z_{k-1}^3\right)=0 \qquad (19)$$

As we see, the conditions for every layer are highly non-linear ones. Transforming (19) we obtain:

$$2\sin 2(\theta_k-\varphi)\begin{Bmatrix} \varepsilon_1^2\left[-U_2-4U_3\cos 2(\theta_k-\varphi)\right]+ \\ +\varepsilon_2^2\left[U_2-4U_3\cos 2(\theta_k-\varphi)\right]+8\varepsilon_1\varepsilon_2U_3\cos 2(\theta_k-\varphi) \end{Bmatrix}\frac{1}{2}\left(z_k-z_{k-1}\right)+$$
$$+2\sin 2(\theta_k-\psi)\begin{Bmatrix} k_1^2\left[-U_2-4U_3\cos 2(\theta_k-\psi)\right]+ \\ +k_2^2\left[U_2-4U_3\cos 2(\theta_k-\psi)\right]+8k_1k_2U_3\cos 2(\theta_k-\psi) \end{Bmatrix}\frac{1}{6}\left(z_k^3-z_{k-1}^3\right)=0 \qquad (20)$$

and finally, dividing by $-\left(z_k-z_{k-1}\right)$:

$$\sin 2(\theta_k-\varphi)\left(\varepsilon_1-\varepsilon_2\right)\left[U_2\left(\varepsilon_1+\varepsilon_2\right)+4U_3\left(\varepsilon_1-\varepsilon_2\right)\cos 2(\theta_k-\varphi)\right]+$$
$$+\sin 2(\theta_k-\psi)\left(k_1-k_2\right)\left[U_2\left(k_1+k_2\right)+4U_3\left(k_1-k_2\right)\cos 2(\theta_k-\psi)\right]\frac{1}{3}\left(z_k^2-z_kz_{k-1}+z_{k-1}^2\right)=0 \qquad (21)$$

As we see, the explicit dependence on the layer z location comes through the last multiplier of the second term.

If we make a sum of (20) for all k and keep in mind (11)-(15), we obtain the united condition

$$\left[A_{16}^{pr.str.}\varepsilon_1^2-A_{26}^{pr.str.}\varepsilon_2^2-\left(A_{16}^{pr.str.}-A_{26}^{pr.str.}\right)\varepsilon_1\varepsilon_2\right]+\left[D_{16}^{pr.cur.}k_1^2-D_{26}^{pr.cur.}k_2^2-\left(D_{16}^{pr.cur.}-D_{26}^{pr.cur.}\right)k_1k_2\right]=0 \qquad (22)$$

or, after using definitions of the stiffness matrices A and D, the relation with clear physical meaning

$$N_{xy}^{pr.str.}(\varepsilon_1-\varepsilon_2)+M_{xy}^{pr.cur.}(k_1-k_2)=0 \qquad (23)$$

The relation (23) is a linear combination of the optimality conditions (20) or (21) and, hence, also it is the optimality condition uniting ones for every layer. The first term in (23), if considered separately as equal to zero, means that the non-linear mid-plane strain tensor is co-axial with the stress flow tensor (or, for $\varepsilon_1 \neq \varepsilon_2$, the shear flow in principal strain axes is equal to zero). The second term in (23), if considered separately as equal to zero, means that the tensor of curvatures is co-axial with the moment tensor (or, for $k_1 \neq k_2$, the twisting moment in principal curvature axes is equal to zero).

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It should be noted that the result in case of orthotropic plate with point-wise orthotropy orientation follows from (23) with the number of layers N=1.

In case of plate-wise variation of layer fiber angles the first order necessary optimality conditions for every layer are different from (21). The conditions in the case mean that the integral of (20) or (21) over the plate is equal to zero. A generalization of (23) for the case will be

$$\int_S dxdy\left[N_{xy}^{pr.str.}(\varepsilon_1-\varepsilon_2)+M_{xy}^{pr.cur.}(k_1-k_2)\right]=0 \qquad (24)$$

where S is the plate mid-plane surface.

3. Examples

There are several simple examples of composite plates satisfying the above indicated tensor co-axiality conditions. The examples may be used as benchmarks for the lay-up optimization software. In all the examples the material axes, the mid-plane principal strain axes and the principal curvature axes are collinear. The four tensors enumerated at the end of the previous Section are co-axial. Note that derivation of the optimality conditions in polar coordinates is analogous to the considered Cartesian coordinate case.
The examples confirm the intuitive understanding of the optimal lay-up, they are:

- Very long plate in y direction, uniformly compressed at the longer sides. The boundary conditions are clamped or simple support ones at the both sides. All sections orthogonal to the y axis are deformed independently of y.
- Uniformly compressed circular plate with cylindrical orthotropy w.r.t. its center. The boundary conditions are clamped or simple support ones. The plate is deformed with cylindrical symmetry.
- Uniformly compressed annular plate with cylindrical orthotropy w.r.t. its geometrical center. The compression stress flow is p_1 at the circular side of radius R_1 and p_2 at the circular side of radius R_2, $R_2 > R_1$. The boundary conditions are clamped or simple support ones at both circular sides. The plate is deformed with cylindrical symmetry.

4. Conclusions

The following conclusions are made for lay-up stiffness optimization of anisotropic composite plates in post-buckling:

- First order necessary optimality conditions are highly non-linear ones;
- The principal mid-plane strain lines and the principal curvature lines play an important role in the conditions;
- The united optimality condition contains two terms, one of them corresponds to mid-plane non-linear strains, another one corresponds to plate bending;
- In the united point-wise optimality condition the term corresponding to mid-plane deformations is equal to zero when the mid-plane strain tensor is co-axial with the mid-plane stress flow tensor (or, for non-equal principal strain values the shear flow in principal strain axes is equal to zero);
- In the united point-wise optimality condition the term corresponding to bending is equal to zero when the tensor of curvatures is co-axial with the moment tensor (or, for non-equal principal curvature values the twisting moment in principal curvature axes is equal to zero).
- The presented examples confirm the intuitive understanding of the optimal lay-up.

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