DAMAGE REPRESENTATION AND DAMAGE EVOLUTION FOR LINEAR VISCOELASTIC UD COMPOSITE MATERIAL

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Keywords: Damage, Linear viscoelasticity, Correspondence principle, Weibull distribution

Abstract
This paper aims to build a damage model for linear viscoelastic UD (unidirectional) composites. The damage representation for the corresponding elastic UD composites with an array of dispersed matrix cracks was derived from Li's work based on continuum damage mechanics (CDM). The elastic-viscoelastic correspondence principle was used to gain the damage representation for corresponding linear viscoelastic UD composites in Laplace domain. A damage evolution law for linear viscoelastic materials was built both on Weibull distribution of defects and on an assumption that damage only occoured in elastic part. An example of constants strain rate was taken to demonstrate how to apply this model. Time temperature superposition principle (TTSP) used in this model will be obtained in future work.

1. Introduction

Some fibre reinforced composites display time dependent behaviour, since the matrix is viscoelastic. The constitutive equations for this specific material can be written in an integral form according to the well-known Boltzman superposition principle. The damage mechanisms of a viscoelastic material could be quite different from that of an elastic material. Schapery has carried out lots of work on deformation, fracture, and damage of linear/non-linear viscoelastic behaviour of monolithic and composite materials by using the thermodynamics of irreversible processes[1-8]. Kumar [9, 10] has built a continuum damage model for linear viscoelastic composite materials upon Talreja's research in CDM. Zocher [11] has analyzed the stress of a matrix-cracked viscoelastic laminate. Both Kumar and Zocher focused on laminates and failed to identify the damage evolution, which is a gap in CDM for viscoelastic materials.

In this paper, we wish to build damage representation and damage evolution for UD composites, the matrix of which is linear viscoelastic with a view that fatigue tests of composites can then be accelerated by combining TTSP with this damage model.

2. Correspondence Principle

Excerpt from ISBN 978-3-00-053387-7

In the viscoelasticity literature, the viscoelastic correspondence principle (CP) is normally referred to corresponding relationship between linear elastic problem and Carson's transformed linear viscoelastic problem.

For elastic materials, the constitutive equation is a linear relationship between stress and strain, and the generalized Hooke's law relating stresses to strains can be written in contracted notation [12] as

$$\sigma_i = C_{ij}\epsilon_j \quad i,j = 1,\dots,6 \tag{1}$$

Where σ_i are the stress components, C_{ij} is the stiffness matrix, and ϵ_j are the strain components. For a linear viscoelastic and non-aging material, the constitutive equations can be written in an integral form based on the well-known Boltzman superposition principle as

$$\sigma_{ij} = \int_0^t C_{ijkl}(t-\tau)\frac{\partial \epsilon_{kl}(\tau)}{\partial \tau}d\tau \tag{2}$$

Or, inversely

$$\epsilon_{ij} = \int_0^t S_{ijkl}(t-\tau)\frac{\partial \sigma_{kl}(\tau)}{\partial \tau}d\tau \tag{3}$$

C_{ijkl} and S_{ijkl} are relaxation modulus tensor and creep compliance tensor, respectively. Take Laplace transform for both sides of (2) and (3) and apply the convolution theorem, we get

$$\bar{\sigma}_{ij} = \bar{C}_{ijkl}\cdot\overline{\left(\frac{\partial \epsilon_{kl}(\tau)}{\partial \tau}\right)} = \bar{C}_{ijkl}\cdot(-\epsilon_{kl}(0) + s\bar{\epsilon}_{kl}) = \tilde{C}_{ijkl}\bar{\epsilon}_{kl} \tag{4}$$

$$\bar{\epsilon}_{ij} = \bar{S}_{ijkl}\cdot\overline{\left(\frac{\partial \sigma_{kl}(\tau)}{\partial \tau}\right)} = \bar{S}_{ijkl}\cdot(-\sigma_{kl}(0) + s\bar{\sigma}_{kl}) = \tilde{S}_{ijkl}\bar{\sigma}_{kl} \tag{5}$$

Where the Laplace transform $\bar{f}(s)$ of a function $f(t)$ is defined as

$$\bar{f}(s) \equiv \mathcal{L}\{f(t)\} \equiv \int_0^\infty e^{-st}f(t)dt \tag{6}$$

and $\tilde{C}_{ijkl}$, the Carson transform of C_{ijkl}, is defined as $\tilde{C}_{ijkl} \equiv s\bar{C}_{ijkl}$ and $\tilde{S}_{ijkl} \equiv s\bar{S}_{ijkl}$.

After taken the Laplace transformation, the integral constitutive equations transform to purely algebraic equations in the Laplace domain. (4) and (5) are analogous to the linear elastic constitutive equations except that they now relate to Laplace transformed stresses and strains. The constitutive equation of a linear viscoelastic material is time dependent. Since the Laplace transformation affects time but not spatial parameters, the corresponding viscoelastic operators obey analogous relations in the Laplace domain [13]. If we obtain the solution for an elastic problem, then we can get the solution for the corresponding viscoelastic problem by replacing all the material properties appearred in the elastic solution by their Carson transforms. The solution is thus obtained in the Laplace domain and needs to be inverted to obtain the time domain solution. This is a well-known correspondence principle of linear viscoelasticity theory [14].

3. Damage Representation

3.1. Damage Representation for Elastic UD Composites

Based on continuum damage mechanics (CDM), Li [15] obtained an expression for stiffness modulus of UD composites with matrix cracks. In his work, the following three assumptions are commonly employed explicitly or implicitly in damage theories:

Excerpt from ISBN 978-3-00-053387-7

(1) The virgin material is homogeneous so that the heterogeneity between reinforcing fibres and the matrix will be neglected;
(2) The virgin material is transversely isotropic. The damage to it takes a form of a single array of cracks, small in size but large in number, with a common orientation such that the damaged material demonstrates an orthotropic behaviour effectively;
(3) The matrix cracks concerned are all mathematical cracks under an unloaded condition and the material around the cracks is free from any initial stresses.

The Compliance of UD composites can be expressed as

$$[S^0] = \begin{bmatrix} 1/E_1^0 & & & & & \\ -\nu_{12}^0/E_1^0 & 1/E_2^0 & & & Symm. & \\ -\nu_{12}^0/E_1^0 & -\nu_{23}^0/E_2^0 & 1/E_2^0 & & & \\ 0 & 0 & 0 & 1/G_{23}^0 & & \\ 0 & 0 & 0 & 0 & 1/G_{12}^0 & \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12}^0 \end{bmatrix} \tag{7}$$

Where $G_{23}^0 = \frac{E_2^0}{2(1+\nu_{23}^0)}$.

For the particular damage mode, matrix cracks have a common orientation with crack surfaces parallel to the fibres direction (axis 1). The damage is described by the stiffness reduction in direction 2. Then

$$E_2 = E_2^0(1-D) \tag{8}$$

Then the compliance with the damage of matrix cracks in direction 2 is

$$[S] = \begin{bmatrix} 1/E_1^0 & & & & & \\ -\nu_{12}^0/E_1^0 & 1/[E_2^0(1-D)] & & & Symm. & \\ -\nu_{12}^0/E_1^0 & -\nu_{23}^0/E_2^0 & 1/E_2^0 & & & \\ 0 & 0 & 0 & 1/\left[G_{23}^0\left(1-\frac{1}{2(1+\nu_{23}^0)}D\right)\right] & & \\ 0 & 0 & 0 & 0 & 1/G_{12}^0 & \\ 0 & 0 & 0 & 0 & 0 & 1/[G_{12}^0(1-kD)] \end{bmatrix} \tag{9}$$

The UD composites become orthotropic after undergoing damage and their properties can be derived from engineering constants of undamaged UD composites and damage variable a
s

$$\begin{gathered} E_1 = E_1^0 \quad E_2 = E_2^0(1-D) \quad E_3 = E_3^0\ (= E_2^0) \\ \nu_{12} = \nu_{12}^0 \quad \nu_{13} = \nu_{13}^0\ (= \nu_{12}^0) \quad \nu_{32} = \nu_{32}^0\ (= \nu_{23}^0) \\ G_{13} = G_{13}^0\ (= G_{12}^0) \quad G_{23} = G_{23}^0\left(1-\frac{1}{2(1+\nu_{23}^0)}D\right) \quad G_{12} = G_{12}^0\ (1-kD) \end{gathered} \tag{10}$$

3.2. Damage Representation for viscoelastic UD Composites

The damage representation of elastic UD composites can be extended to the viscoelastic case by replacing the engineering constants by the Carson transformation of corresponding creep compliance.

$$[\tilde{S}] = \begin{bmatrix} 1/\tilde{E}_1^0 & & & & & \\ -\tilde{\nu}_{12}^0/\tilde{E}_1^0 & 1/\left[\tilde{E}_2^0(1-\hat{D})\right] & & & Symm. & \\ -\tilde{\nu}_{12}^0/\tilde{E}_1^0 & -\tilde{\nu}_{23}^0/\tilde{E}_2^0 & 1/\tilde{E}_2^0 & & & \\ 0 & 0 & 0 & 1/\left[\tilde{G}_{23}^0\left(1-\frac{1}{2(1+\tilde{\nu}_{23}^0)}\hat{D}\right)\right] & & \\ 0 & 0 & 0 & 0 & 1/\tilde{G}_{12}^0 & \\ 0 & 0 & 0 & 0 & 0 & 1/\left[\tilde{G}_{12}^0(1-k\hat{D})\right] \end{bmatrix} \tag{11}$$

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Where $\widehat{D} = 1 - \tilde{E}_2/\tilde{E}_2^0$.
If we obtain the relaxation moduli of undamaged viscoelastic UD composites and apply Carson's transformation, then we can get the damage representation of viscoelastic UD composites in Laplace domain. Damage variable $\widehat{D}$ is controlled by damage evolution law and k can be gained from experiments.

4. Damage Evolution Law

4.1. Damage Evolution for Linear Elastic Materials

After obtaining damage representation of linear viscoelastic composites, we can further consider how damage evolves. Like damage representation, the damage evolution can be started from elastic case.

Suppose there is a representative volume element (RVE) in a UD composite. There are a large number of different size/level defects in this RVE and the sizes/levels of the defects are described by the sensitivity of effective stress which can develop the defects into damage. The probability density of these defects can be described by the Weibull distribution.

$$d(\sigma_e; \lambda, h) = \begin{cases} \frac{h}{\lambda}\left(\frac{\sigma_e}{\lambda}\right)^{h-1} e^{-\left(\frac{\sigma_e}{\lambda}\right)^h} & \sigma \geq 0 \\ 0 & \sigma < 0 \end{cases} \tag{12}$$

Where h and λ are constants, effective stress is defined as

$$\sigma_e = \frac{\sigma}{1 - D} \tag{13}$$

Then

$$D = \int_0^{\sigma_e} d\,\mathrm{d}x = 1 - e^{-\left(\frac{\sigma_e}{\lambda}\right)^h} \tag{14}$$

(14) can be the damage evolution law for elastic case.

4.2. Damage Evolution for Linear Viscoelastic Materials

Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. The most validated mathematical model to describe viscoelasticity in 1 dimensional is the Wiechert model

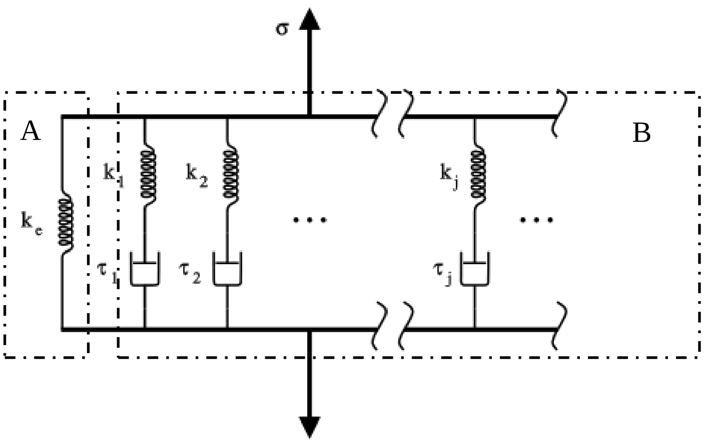

Figure 1: The Wiechert model.

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And the relaxation modulus for this model is

$$E_{rel}(t) = k_e + \sum_j k_j \, exp\left(-\frac{t}{\tau_j}\right) \tag{15}$$

During deformation, the energy release in viscoelastic materials can be divided into two parts, one is energy released in developping new surface in elastic part, the other is energy dissipated in the form of heat in viscous part.

We assume in this paper that the damage only happen in elastic part (part A). When the effective stress applied on spring k_e is beyond the limit, the damage occurs. When effective stress applied on k_e increases, the corresponding damage evolves. When the spring k_e fails, the whole system will fail. Then the relaxation modulus with damage can be defined as

$$E_D(t) = (1-D)k_e + \sum_j k_j \, exp\left(-\frac{t}{\tau_j}\right) \tag{16}$$

Where $E_D(t)$ can be $E_2(t)$ (transverse modulus) in our damage representation in 3 dimension and D can be substituted by (14). Then

$$\tilde{E}_2 = e^{-\left(\frac{k_e \epsilon}{\lambda}\right)^h} k_e + \sum_j \frac{k_j s}{\left(s + \frac{1}{\tau_j}\right)} \tag{17}$$

Then

$$\widehat{D} = 1 - \frac{\tilde{E}_2}{\tilde{E}_2^0} = \left[1 - e^{-\left(\frac{k_e \epsilon}{\lambda}\right)^h}\right] k_e \Bigg/ k_e + \sum_j \frac{k_j s}{\left(s + \frac{1}{\tau_j}\right)} \tag{18}$$

Where ϵ can be the function of time t. We can put $\widehat{D}$ back to (11) to get the complete damage model for linear viscoelastic UD composites.

The stress applied to a viscoelastic material can be expressed by using the famous Boltzman superposition principle as

$$\begin{aligned} \sigma(t) &= E_D(t)\epsilon_0 + \int_0^t E_D(t-\tau) \frac{\partial \epsilon(\tau)}{\partial \tau} \mathrm{d}\tau \\ &= (1-D)k_e \left[\epsilon_0 + \int_0^t \frac{\partial \epsilon(\tau)}{\partial \tau} \mathrm{d}\tau\right] \\ &+ \sum_j k_j \, exp\left(-\frac{t}{\tau_j}\right)\epsilon_0 + \int_0^t \sum_j k_j \, exp\left(-\frac{t-\tau}{\tau_j}\right) \frac{\partial \epsilon(\tau)}{\partial \tau} \mathrm{d}\tau \end{aligned} \tag{19}$$

$(1-D)k_e\left[\epsilon_0 + \int_0^t \frac{\partial \epsilon(\tau)}{\partial \tau} \mathrm{d}\tau\right]$ should be the stress applied to the elastic part, then $k_e\left[\epsilon_0 + \int_0^t \frac{\partial \epsilon(\tau)}{\partial \tau} \mathrm{d}\tau\right]$ is the effective stress in the elastic part and the damage evolution is controlled by $k_e\left[\epsilon_0 + \int_0^t \frac{\partial \epsilon(\tau)}{\partial \tau} \mathrm{d}\tau\right]$.

We can take constant strain rate test for example. If a static test for viscoelastic material is controlled by constant strain rate a, then (19) will transform into

$$\sigma(t) = (1-D)ak_e t + a\left[\sum_j k_j \tau_j - \sum_j k_j \tau_j \, exp\left(-\frac{t}{\tau_j}\right)\right] \tag{20}$$

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And the damage variable D from (14) will be

$$D = 1 - e^{-\left(\frac{k_e \epsilon}{\lambda}\right)^h} = 1 - e^{-\left(\frac{a k_e t}{\lambda}\right)^h} \tag{21}$$

Then (21) can be substituted into (20) to obtain stress σ with damage evolution consideration duly incorporated

$$\sigma\,(t) = e^{-\left(\frac{a k_e t}{\lambda}\right)^h} a k_e t + a\left[\sum_j k_j \tau_j - \sum_j k_j \tau_j \, exp\left(-\frac{t}{\tau_j}\right)\right] \tag{22}$$

The constants k and λ in Weibull distribution are specific properties of certain materials which can be obtained from experiment. Then damage evolution with certain constant strain rate is controlled by time t.

5. Application

In this paper, we gained both analytical solution for damage variable D and damage-evolution-varying stress. However, it is not explicit enough for us to understand how damage evolves. We can bring in specific viscoelastic properties to get a clear picture of damage evolution. Thus we introduce viscoelastic properties of composite from exist paper [10], in which the viscoelastic properties of laminate composite were described by a Prony series (Wiechert model) as table 1.

Table 1 Prony series representation of relaxation modulus

		E_{rel} (GPa)	
j	k_j	τ_j	$k_e = 154.06$
1	0.0436	0.0009	
2	0.0478	0.0096	
3	0.1011	0.2022	
4	0.2087	4.174	
5	0.4125	82.49	
6	0.7387	1447	
7	1.162	23250	
8	1.266	253200	
9	1.284	2568000	
10	0.3801	7602000	
11	1.081	216300000	

For the purpose of illustration, h and λ are given the value of 0.5 and 8 respectively. Damage evolution curve is described by (21). Then damage variable D varying with strain ϵ can be illustrated as Figure 2.

From (21) and Figure 2, we can find that damage evolution is controlled by strain. It should be noted that failure here is defined as a situation in which the damage variable D is approaching to 1. The stress applied in elastic part is described by the first part of (22) as

$$\sigma_{k_e} = e^{-\left(\frac{k_e a t}{\lambda}\right)^h} k_e a t = e^{-\left(\frac{k_e \epsilon}{\lambda}\right)^h} k_e \epsilon \tag{23}$$

which is also controlled by strain. Then the stress applied in elastic part varying with strain can be illustrated as Figure 3.

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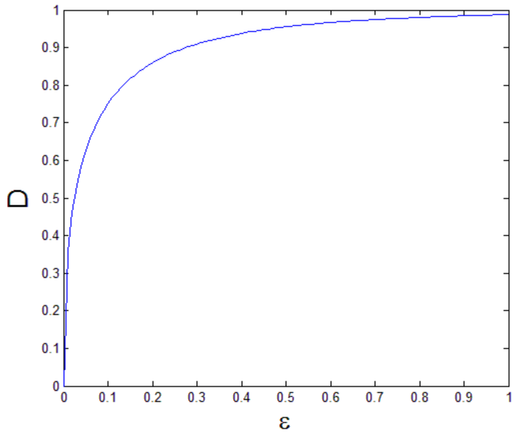

Figure 2 Damage variable varies with strain

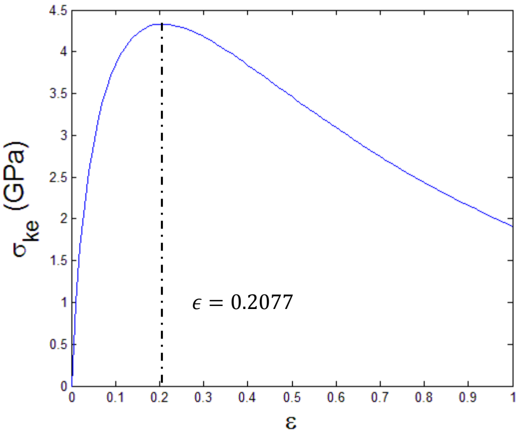

Figure 3 Stress applied on elastic part varies with strain

The extremum can be gained from

$$\frac{\mathrm{d}\sigma_{k_e}}{\mathrm{d}\epsilon} = 0 \tag{24}$$

In this specific example, we can get that stress applied in elastic part will reach the maximum stress when $\epsilon = 0.2077$. Then the failure of whole material can be defined in this strain.

We can set three loading rates at 1mm/min, 0.01mm/min and 0.0001mm/min, and the test time are correspondingly 0.2077min, 20.77min and 2077min. Then all of three tests have same strain and same damage evolution. Stress strain curves of all of three tests are expressed as Figure 4.

From Figure 4 we can see the failure stress (corresponding strain is 0.2077) is getting smaller when loading rate is getting smaller. Then a curve of failure stress varying with testing time can be made as Figure 5.

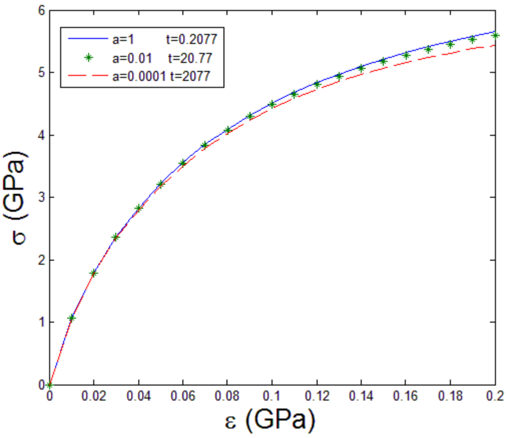

Figure 4 stress strain curve with different loading rates

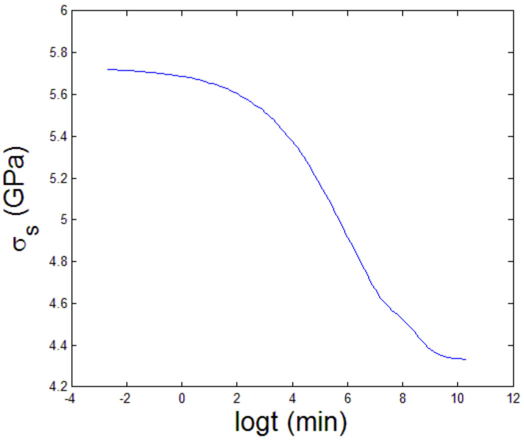

Figure 5 Failure stress varying with testing time

In the test of constants strain rate of linear viscoelastic composites, different loading rates will lead to different test time. The lower loading rate is, the longer test time will be. If we want to obtain the information of failure stress and testing time from experiments, it could be a time-consuming, high-cost and even impossible task. However, from Figure 5 we can directly acquire the failure stress at any test time. The details of Figure 5 are controlled by constants k_e, h and λ. So we can imagine that if we get value of h and λ from experiments, we can make more accurate prediction of damage evolution and failure stress at different loading rates. Miyano gained the similar curve of failure stress versus time by using time-temperature superposition principle, and we can also put the temperature effects into our damage model and use the time-temperature superposition principle in the future.

6. Conclusion

Firstly, A CDM based damage representation based on Li's work is presented in this paper to model the effects of a specific and fixed level of damage in the linear viscoelastic UD composites. Then the damage evolution law of elastic case was obtained from the Weibull distribution of sensitivity of defects probability density to effective stress. Finally, the damage evolution law of linear viscoelasticity was gained from assumptions that linear viscoelasticity was described by the Wiechert model and the damage only evolves in elastic part. At last an example of constants strain rate was put into this model to check the feasibility of this damage model. Temperature effects on this model will be discussed in future work.

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