VARIATIONAL ANALYSIS OF ANGLE-PLY LAMINATES WITH ARRAYS OF PARALLEL INTRALAMINAR AND DELAMINATION CRACKS

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Abstract
Effective thermoelastic properties and stress fields in angle-ply laminates with intralaminar matrix cracks depend on the crack spatial distribution. Most of the existing approaches for stress analysis of a cracked laminate are either limited to cross-ply laminates or to symmetrical (and periodic) arrays of cracks. However, in many cases laminates form antisymmetric or staggered microcrack arrays. A new variational stress analysis is suggested, when all the component of the stress tensor are assumed to be functions of the transverse direction as well. The in-plane stresses are assumed to be linear functions of the transverse coordinate. An admissible stress field is derived that satisfies equilibrium, boundary and traction continuity conditions. Using the principle of minimum complementary energy an optimal stress field is evaluated. The new analysis is applicable to any angle-ply laminate with parallel (but not necessarily coplanar) intralaminar cracks subjected to in-plane membrane forces and moments.

1. Introduction

The variational approach was originated by Hashin [1] for an approximate stress field in a symmetric cross-ply $[0_n/90_m]_S$ with intralaminar cracks in the transverse ply and the lower bound for the effective elastic moduli of the laminate. The main idea of the approach consists of developing a stress state that satisfies all equilibrium, boundary and traction continuity conditions, and minimizes the complimentary energy of the laminate in attempt to have the best approximation for the stresses. It has been widely used and extended to different cases. One can refer to work by Nairn, who has extensively used the variational analysis, extended it to thermal loading, and used it with an energy based fracture criterion to describe crack accumulation in cross-ply laminates (see, e.g. [2] and references therein). Li and Hafeez [3] extended the approach to a laminate with more than three plies. Vinogradov and Hashin [4] have looked at an angle-ply laminate with cracks in the middle ply. Recently, the method has gained more attention, e.g. [5,6]. All this work deals with symmetric laminates and symmetric arrays of transverse cracks.

In the present paper we address the case when laminates of arbitrary symmetry form antisymmetric or staggered microcracks. To our best knowledge, the only work that has addressed this problem is [7], where a $[90_m/0_n]_S$ cross-ply having cracks in the outer plies was considered under uniaxial tension. Antisymmetry, associated with staggered crack arrangements was allowed for by splitting the 0° ply into two to allow local bending moments and keeping the original assumptions in [1] of axial stress being a function of one coordinate. The approach for admissible stress state developed in the present paper assumes that the in-plane stresses in the cracked laminate are functions of two coordinates, x in the direction normal to the crack surfaces and z in the transverse direction. The method can be used with arbitrary layups, membrane loads and moments, and any array of parallel intralaminar cracks.

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2. Admissible Stress Field

Consider an N-ply laminate sample in the xy plane under a uniform in-plane membrane forces N_x, N_y, N_{xy} and moments M_x, M_y, M_{xy}. We next impose an arbitrary state of damage in a family of plies defined by a certain fibre orientation θ^*, when the intralaminar cracks are parallel to the fibre direction, but are not necessarily coplanar. One can then rotate the coordinate system by $(\pi/2 - \theta^*)$ about the z-axis, such that the crack surfaces are all normal to the x-axis and the cracked plies become 90° plies (Fig. 1).

Compacted index notations are used, which are common in the classical laminate theory. For arbitrary orientation of plies in the laminate, when there are no cracks the in-plane stresses $\sigma_1^{0(m)} \equiv \sigma_{xx}^{0(m)}$, $\sigma_2^{0(m)} \equiv \sigma_{yy}^{0(m)}$ and $\sigma_6^{0(m)} \equiv \sigma_{xy}^{0(m)}$ in any ply (m) are linear function of the transverse coordinate z, which can be obtained from a simple analysis using the classical laminate theory, and the rest of the stress tensor components $\sigma_4^0 \equiv \sigma_{yz}^0$, $\sigma_5^0 \equiv \sigma_{xz}^0$ and $\sigma_3^0 \equiv \sigma_{zz}^0$ are equal to zero. The cracks introduce stress perturbations, which are denoted $\sigma_i^{(m)}$, where i ranges over 1 to 6.

Let us represent the stresses in the m-th ply of the cracked material as a superposition of the stresses in the uncracked material and perturbation stresses due to the presence of the cracks

$$\sigma_i^{C(m)}(x,z) = \sigma_i^{0(m)}(z) + \sigma_i^{(m)}(x,z). \tag{1}$$

All the previous versions of the stress-based versions of the variational analysis are based on the assumption that the perturbation in-plane stresses are functions of x-coordinate only. This assumption limited the analysis to the case of symmetrical laminates and symmetrical geometry of intralaminar cracks. For a generic angle-ply, the perturbation stresses must satisfy the zero traction boundary condition on the crack faces. This implies that the perturbation stresses contributing to the traction vectors in the cracked plies on the crack surfaces must be linear functions of z-coordinate.

We will now assume that the in-plane perturbation stresses $\sigma_1^{(m)}$, $\sigma_2^{(m)}$ and $\sigma_6^{(m)}$ in ply m are linear functions of z everywhere in the cracked laminate, i.e. for every coordinate x. This is a minimum requirement that would allow us to analyse stresses in angle-ply laminates or laminates with non-symmetric arrays of transverse cracks. According to this assumption the in-plane perturbation stresses can be expressed in the form

$$\sigma_i^{(m)}(x,z) = \phi_i^{(m)}(x) + \psi_i^{(m)}(x)\,\zeta, \qquad i = 1,2,6, \tag{2}$$

where ζ is the dimensionless z-coordinate, defined as

$$\zeta = (z - \sum_{j=1}^{m-1} t_j)\,/t_m \tag{3}$$

and running from 0 to 1 within each ply. Functions $\phi_i^{(m)}(x)$ and $\psi_i^{(m)}(x)$ are yet unknown functions to be found.

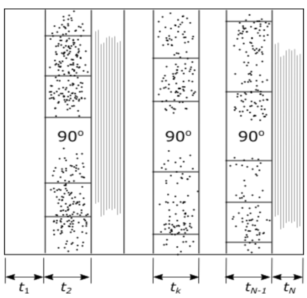

Figure 1: Schematic presentation of a cracked laminate in the rotated coordinate system.

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The resulting stresses must satisfy equilibrium and all interface and boundary conditions, forming so-cold admissible stress field. Equilibrium differential equations written for the perturbation stresses using the non-dimensional coordinates x and ζ:

$$\begin{aligned} &\frac{\partial}{\partial x}\sigma_1^{(m)}(x,\zeta)+\frac{1}{\lambda_m}\frac{\partial}{\partial \zeta}\sigma_5^{(m)}(x,\zeta)=0,\\ &\frac{\partial}{\partial x}\sigma_6^{(m)}(x,\zeta)+\frac{1}{\lambda_m}\frac{\partial}{\partial \zeta}\sigma_4^{(m)}(x,\zeta)=0,\\ &\frac{\partial}{\partial x}\sigma_5^{(m)}(x,\zeta)+\frac{1}{\lambda_m}\frac{\partial}{\partial \zeta}\sigma_3^{(m)}(x,\zeta)=0.\end{aligned} \tag{4}$$

where $\lambda_m = t_m/t_0$, t_m is the thickness of the m-th ply and t_0 is an arbitrary normalization thickness, which for a cross-ply containing one cracked ply is usually chosen as the thickness of the cracked ply.

Substituting the linear expression for the perturbation into the equilibrium equations and then integrating the equations with respect to ζ gives the following general expressions for the out-of-plane stresses:

$$\begin{aligned} \sigma_5^{(m)}(x,\zeta) &= -\lambda_m\,\phi_1'^{(m)}(x)\zeta-\lambda_m\,\psi_1'^{(m)}(x)\frac{\zeta^2}{2}-g_5^{(m)}(x),\\ \sigma_4^{(m)}(x,\zeta) &= -\lambda_m\,\phi_6'^{(m)}(x)\zeta-\lambda_m\,\psi_6'^{(m)}(x)\frac{\zeta^2}{2}-g_4^{(m)}(x),\\ \sigma_3^{(m)}(x,\zeta) &= \lambda_m^2\,\phi_1''^{(m)}(x)\frac{\zeta^2}{2}+\lambda_m^2\,\psi_1''^{(m)}(x)\frac{\zeta^3}{6}+g_5'^{(m)}(x)\zeta+g_3^{(m)}(x),\end{aligned} \tag{5}$$

where $g_3(x)$, $g_4(x)$ and $g_5(x)$ are integration functions. These integration functions can be expressed in terms of the functions $\phi_i(x)$ and $\psi_i(x)$ using the traction-free condition at the external surfaces of the first ply

$$\sigma_3^{(1)}(x,0)=\sigma_4^{(1)}(x,0)=\sigma_5^{(1)}(x,0)=0, \tag{6}$$

and traction continuity conditions at the interfaces between the plies

$$\begin{aligned} \sigma_3^{(m)}(x,0) &= \sigma_3^{(m-1)}(x,1),\\ \sigma_4^{(m)}(x,0) &= \sigma_4^{(m-1)}(x,1),\\ \sigma_5^{(m)}(x,0) &= \sigma_5^{(m-1)}(x,1).\end{aligned} \tag{7}$$

One can start with the external surface of the first ply and iteratively progress towards the opposite external surface to evaluate all the integration functions. Eq. 6 and Eqns. 7 applied to $(N-1)$ interfaces between the N plies provide the required constraints for all the unknown integration functions $g(x)$.

Similar to Eq. 6 the traction-free condition at the external surface of the last ply can be written as

$$\sigma_3^{(N)}(x,1)=\sigma_4^{(N)}(x,1)=\sigma_5^{(1)}(x,1)=0 \tag{8}$$

However, it can be shown that these zero-traction conditions are satisfied automatically if the perturbation membrane forces and moments evaluated using the linear perturbation in-plane stresses are required to be zero, meaning that the forces and moments applied to the laminate do not change due to the presence of cracks. For the perturbation membrane forces we have

$$\sum_1^N \lambda_m \int_0^1 \sigma_i^{(m)}(x,\zeta)\,d\zeta = 0, \qquad i=1,2,6 \tag{9}$$

and for perturbation moments

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$$\sum_{1}^{N} \lambda_m^2 \int_0^1 \sigma_i^{(m)}(x,\zeta)\,\zeta\, d\zeta = 0, \qquad i = 1,2,6 \tag{10}$$

In order to proceed with the solution, we express the stresses in a matrix form

$$\boldsymbol{\sigma}^{(m)}(x,\zeta) = \boldsymbol{A}_0^{(m)}(\zeta)\,\boldsymbol{f}(x) + \boldsymbol{A}_1^{(m)}(\zeta)\,\boldsymbol{f}'(x) + \boldsymbol{A}_2^{(m)}(\zeta)\,\boldsymbol{f}''(x), \tag{11}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)^T$ is a (6×1) vector of the stress tensor components, $\boldsymbol{f}(x)$ is a $(6N \times 1)$ vector of unknown functions:

$$\boldsymbol{f}(x) = \left(\phi_1^{(1)}(x), \phi_2^{(1)}(x), \phi_6^{(1)}(x), \psi_1^{(1)}(x), \psi_2^{(1)}(x), \psi_6^{(1)}(x), \phi_1^{(2)}(x), \phi_2^{(2)}(x), \dots, \psi_6^{(N)}(x)\right)^T \tag{12}$$

and the matrices $\boldsymbol{A}_0^{(m)}$, $\boldsymbol{A}_1^{(m)}$, $\boldsymbol{A}_2^{(m)}$ are the ζ-dependant coefficient matrices of the functions of x and their first and second derivatives, respectively. The elements of these matrices are simple polynomials of ζ.

Correspondingly, the six constraints of Eqns. 9 and 10 can be written in the matrix form

$$\boldsymbol{B}\,\boldsymbol{f}(x) = 0, \tag{13}$$

where $\mathbf{B}$ is a $(6 \times 6N)$ matrix. In principle, these constraints can be used to express the 6 functions defining the stress state in the last (or any) ply in terms of the stresses in other $(N-1)$ plies.

3. Complementary Energy Minimization

The equations in the previous section define the admissible stress state for an angle-ply laminate with intralaminar matrix cracks in the family of 90° plies. The stress field satisfy equilibrium, all traction boundary conditions on the external surfaces of the laminate and traction continuity between plies. The stress state is expressed in terms of $6N$ unknown functions of x, when 6 of the unknown functions are dependant by linear constraints of Eq. 13. The unknown functions are now to be evaluated by using the principle of minimum complimentary energy.

We now consider a `unit cell' of the cracked laminate bounded by two adjacent parallel cracks, that may or may not belong to the same ply (Fig. 2), the origin of the *x*-axis is chosen in the middle between the cracks. In fact, this implies that the cracked laminate can be reconstructed using this unit cell by either its translation along the *x*-axis or by translation and mirror-reflection of the cell with respect to the *yz*-plane (staggered arrangement of cracks), see Figre 2. The distance between the two cracks that bound the unit cell is $2a$, normalized with respect to t_0. A variational solution that minimizes complimentary energy is found by minimization of the functional given by

$$U_C = t_0^2 \int_{-a}^{a} \sum_{m=1}^{N} \lambda_m \int_0^1 \sigma_i^{(m)} S_{ij}^{(m)}\, \sigma_j^{(m)} dx\, d\zeta = t_0^2 \int_{-a}^{a} \left(\boldsymbol{f}^p(x)\right)^T \boldsymbol{C}_{pq}\, \boldsymbol{f}^q(x)\, dx, \tag{14}$$

where the superscripts p and q represent the order of the *x*-derivative of the vector function $\boldsymbol{f}(x)$. The matrices $\boldsymbol{C}_{pq}$ are defined as:

$$\boldsymbol{C}_{pq} = \sum_{m=1}^{N} \lambda_m \int_0^1 \left(\boldsymbol{A}_p^{(m)}(\zeta)\right)^T \boldsymbol{S}^{(m)}\, \boldsymbol{A}_q^{(m)}(\zeta) d\zeta \tag{15}$$

It turns out that for monoclinic symmetry of the ply material $\boldsymbol{C}_{01} = \boldsymbol{C}_{10} = \boldsymbol{C}_{21} = \boldsymbol{C}_{12} = 0$.

The problem now is to find functions $\boldsymbol{f}(x)$ that minimize the functional U_C, subject to the linear constraints $\boldsymbol{B}\boldsymbol{f}(x) = 0$. The simplest way of solving this variational problem is to add the linear constraints to the functional, using Lagrange multiplyers:

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$$F = \int_{-a}^{a} [(\boldsymbol{f}^p)^T \boldsymbol{C}_{pq}\, \boldsymbol{f}^q + \boldsymbol{\lambda}^T\, \boldsymbol{B}\, \boldsymbol{f}]\ dx \tag{16}$$

where $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}(x)$ is a (6×1) vector of Lagrange multipliers. When a variation is taken for the functional above, the stationary value condition $\delta F = 0$ leads to the Euler-Lagrange equations as the governing equations for the problem:

$$\begin{bmatrix} \boldsymbol{M}_0 & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \boldsymbol{M}_2 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}'' \\ \boldsymbol{\lambda}'' \end{bmatrix} + \begin{bmatrix} \boldsymbol{M}_4 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^{(iv)} \\ \boldsymbol{\lambda}^{(iv)} \end{bmatrix} = 0, \tag{17}$$

where $\mathbf{M}_0 = \mathbf{C}_{00}$, $\mathbf{M}_2 = \mathbf{C}_{02} + \mathbf{C}_{20} - \mathbf{C}_{11}$ and $\mathbf{M}_4 = \mathbf{C}_{22}$. This augmented formulation can now be written in the form

$$\widetilde{\boldsymbol{M}}_0\, \tilde{\boldsymbol{f}} + \widetilde{\boldsymbol{M}}_2 \tilde{\boldsymbol{f}}'' + \widetilde{\boldsymbol{M}}_4 \tilde{\boldsymbol{f}}^{(iv)} = 0, \tag{18}$$

where the quantities with tilde denote the augmented matrices and vectors.

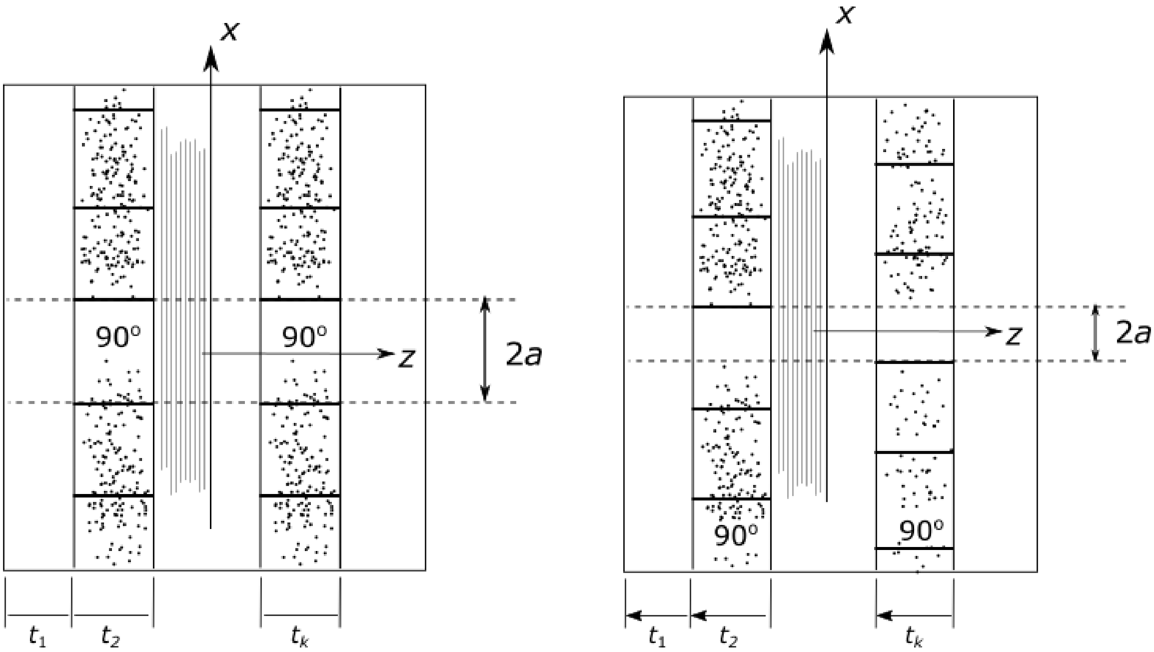

Figure 2: Schematic presentation of a periodic (left) and staggered (right) arrangements of cracks.

Eq. 18 is a system of differential equations with constant coefficients and it can be reduced to the form of an eigenproblem. It can be seen from the structure of the functional, that matrices $\widetilde{\boldsymbol{M}}_4$ and $\widetilde{\boldsymbol{M}}_2$ are singular. In order proceed, we will multiply the equation by the inverse of the matrix $\widetilde{\boldsymbol{M}}_0$:

$$\tilde{\boldsymbol{f}} + \widetilde{\boldsymbol{M}}_0^{-1} \widetilde{\boldsymbol{M}}_2 \tilde{\boldsymbol{f}}'' + \widetilde{\boldsymbol{M}}_0^{-1} \widetilde{\boldsymbol{M}}_4 \tilde{\boldsymbol{f}}^{(iv)} = 0 \tag{19}$$

Expressing the vector $\tilde{\boldsymbol{f}}$ in terms of its second and fourth derivatives and augmenting the matrix equations by the identity $\tilde{\boldsymbol{f}}'' = \mathbf{I}\, \tilde{\boldsymbol{f}}''$, $\mathbf{I}$ being the identity matrix, one can write

$$\begin{bmatrix} \tilde{\boldsymbol{f}} \\ \tilde{\boldsymbol{f}}'' \end{bmatrix} = \begin{bmatrix} -\widetilde{\boldsymbol{M}}_0^{-1} \widetilde{\boldsymbol{M}}_2 & -\widetilde{\boldsymbol{M}}_0^{-1} \widetilde{\boldsymbol{M}}_4 \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{f}}'' \\ \tilde{\boldsymbol{f}}^{(iv)} \end{bmatrix}. \tag{20}$$

Since the vector in the right hand side of the matrix equation above is the second derivative with respect to x of the vector in the left-hand side, we will be looking for a solution in the form of exponential functions:

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$$\tilde{\boldsymbol{f}}'' = \sum_i C_i \exp(r_i x)\, \boldsymbol{u}_i. \tag{21}$$

Here r_i are eigenvalues, $\boldsymbol{u}_i$ are the corresponding eigenvectors and C_i are constants that are to be found using boundary conditions at the crack surfaces. Eqn. 20 reduces to the following eigenproblem for $1/r_i^2$:

$$\left(\begin{bmatrix} -\widetilde{\boldsymbol{M}}_0^{-1}\widetilde{\boldsymbol{M}}_2 & -\widetilde{\boldsymbol{M}}_0^{-1}\widetilde{\boldsymbol{M}}_4 \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} - \frac{1}{r_i^2}\boldsymbol{I} \right) \boldsymbol{u}_i = 0, \tag{22}$$

which can be solved numerically for r_i and u_i using standard algorithms. Notice that the matrix in Eq. 22 is singular and only non-zero eigenvalues $(1/r_i^2)$ should be selected.

4. Boundary Conditions

The physical boundary conditions are given in terms of the zero traction vector on the crack surfaces, defined by coordinates x^*. For the total stresses we need to require that in the cracked plies

$$\sigma_1^C(x^*, z) = \sigma_5^C(x^*, z) = \sigma_6^C(x^*, z) = 0, \tag{23}$$

when for the perturbation stresses at every crack station we can write

$$\begin{aligned} \sigma_1(x^*, z) &= -\sigma_1^0(z), \\ \sigma_5(x^*, z) &= 0, \\ \sigma_6(x^*, z) &= -\sigma_6^0(z). \end{aligned} \tag{24}$$

The first and the third boundary conditions in Eq. 24 define the values of the functions $\phi_1(x^*)$, $\psi_1(x^*)$, $\phi_6(x^*)$, $\psi_6(x^*)$ in the cracked ply that should be equal the negative of the constant and linear terms of the stress functions in the uncracked laminate. The second condition in Eq. 24 requires that $\phi_1'(x^*) = \psi_1'(x^*) = 0$.

These boundary conditions do not suffice to evaluate all the constants C_i in Eq. 21. However, the calculus of variation provides additional conditions for functions that do not have prescribed boundary conditions, usually called "natural" boundary conditions. For the functional in hand, these natural boundary conditions have the form

$$\left(\frac{\partial F}{\partial \boldsymbol{f}'} - \frac{d}{dx}\frac{\partial F}{\partial \boldsymbol{f}''} \right)\Bigg|_{x=x^*} = 0. \tag{25}$$

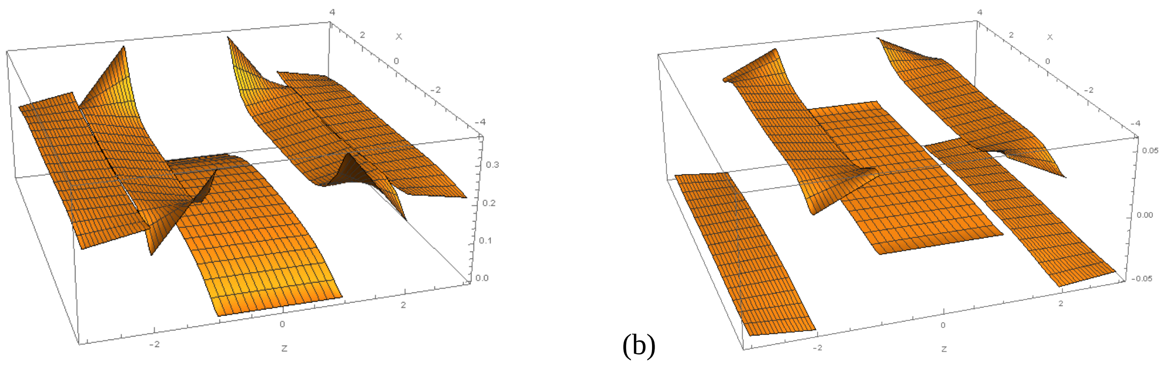
These boundary conditions written for all the unknown functions in the uncracked plies provide the necessary conditions for determination of the unknown constants C_i. In case of staggered microcracks two adjacent fragments can be considered with continuity traction conditions at the interface between the fragments and natural boundary conditions on the outmost cross-sections. Treatment of induced delamination cracks, although is very similar, requires introduction of additional (differential) constraints for zero tractions on delaminated surfaces. The details of the scheme for non-periodic (random) arrays of microcracks and delamination cracks will be reported elsewhere.

5. Numerical Examples

Fig.3 shows stresses in $[\pm 15/90]$s GFRP laminate with cracks in the 90° ply subjected to uniaxial tension in the *x*-direction. Laminates of this type have previously been treated by averaging the off-axis plies to form a three-ply symmetrical scheme [8] or by making the same assumption of z-independent in-plane stresses in all the plies. The present approach allows us to get more detail about the stress fields in the cracked laminate. In Fig. 3 the stress distribution is shown in the *xz*-plane over a unit cell of the normalized length of 8. We can see that the $-15°$ plies are more affected by the presence of cracks in the middle ply; stress concentrations are clearly observed at vicinity of the transverse crack tips, which would definitely be expected.

A similar effect is observed in $[90/0]$s laminate with a staggered arrangement of cracks in the 90° plies under uniaxial tension. Fig. 4 shows the axial stress σ_{xx} in a laminate fragment bounded by two cracks: one in the left 90° ply at $x = 4$ and one in the right 90° ply at $x = -4$. The antisymmetric stress distribution demonstrates significant gradients in the cross-sections coplanar with the transverse cracks.

Substitution of the resultant stresses back into the expression of the complimentary energy and integration over the volume of the unit cell leads to the lower bound on the effective stiffness of the cracked material. Effective stiffness as a function of crack density for $[\pm 15/90_4]$s and $[\pm 30/90_4]$s GFRP laminates are shown in Fig. 5. Excellent agreement with experimental data published by Joffe et.al. [8] is observed.



(a) (b)

Figure 3: Stress distribution in $[\pm 15/90]$s under unidirectional tension: (a) σ_{xx}, (b) σ_{xy}.

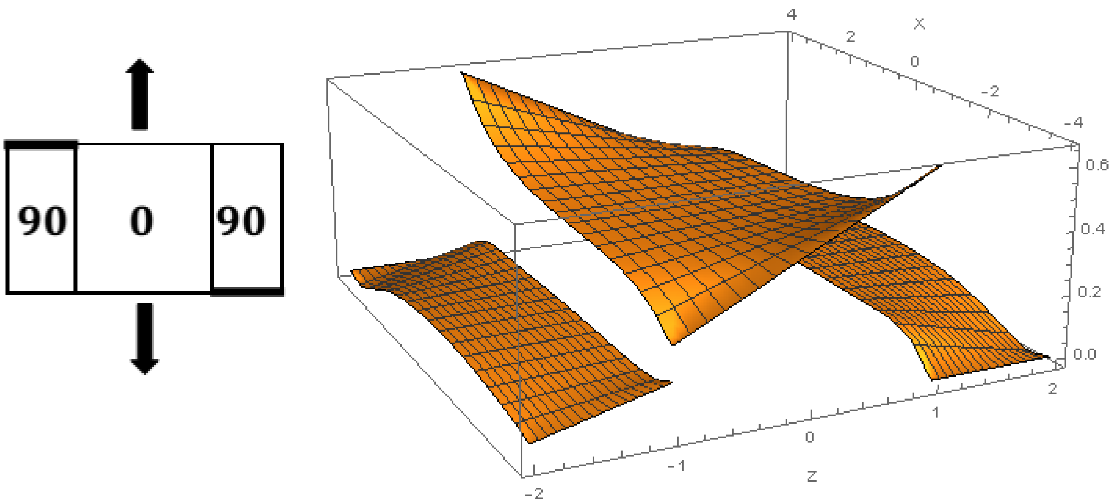


Figure 4: Distribution of σ_{xx} in $[90/0]$s subjected to unidirectional tension with staggered arrangement of cracks.

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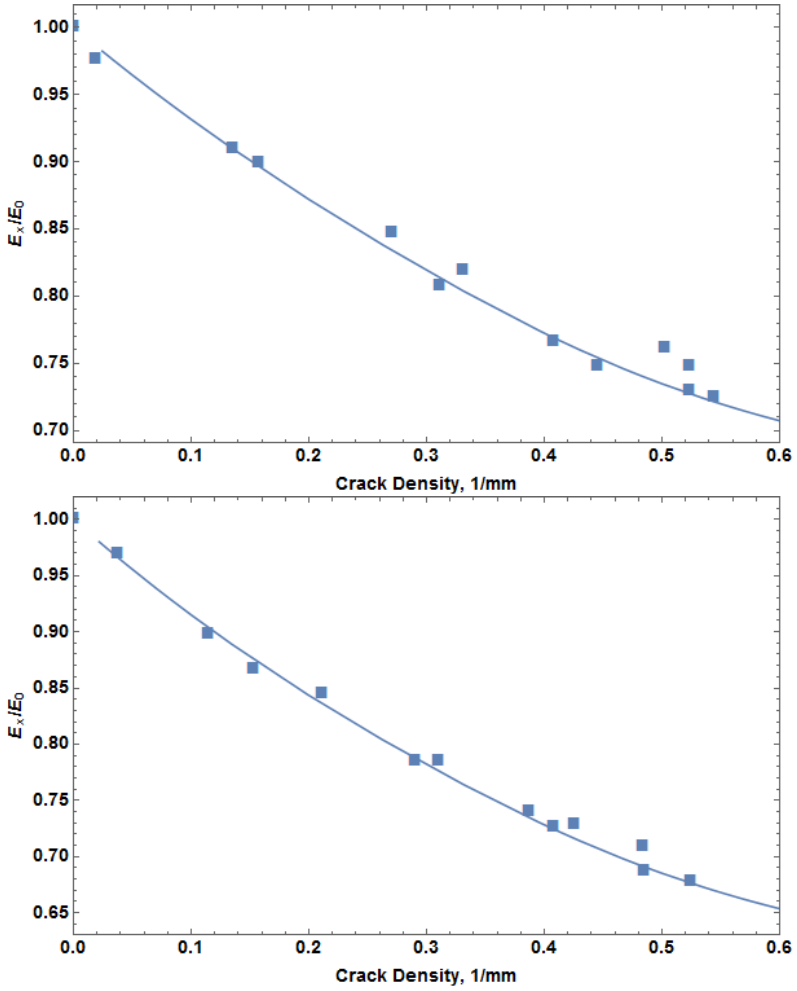

Figure 5: Young modulus reduction as a function of transverse crack density in $[\pm 15/90_4]$s (top) and $[\pm 30/90_4]$s (bottom). Experimental data from [8].

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