Geometrically nonlinear asymptotic homogenization modeling of a thin composite layer with wavy surfaces

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Abstract

The mechanical behavior of composite structures with periodic configurations depend not only on the macroscopic response of the structure, but also on the microscopic characteristics of the various constituents of the reinforcing fibers and matrix materials. This paper develops the geometrically non-linear composite plate model to analyze the effective elastic properties through the application of the modified asymptotic homogenization method. To make the method plausible for a three-dimensional problem, two sets of 'rapid' coordinates, one in the tangential direction associated with the rapid periodic oscillation in the composite properties and geometrical shape of upper and lower surfaces of the plate and the other in the transverse direction corresponding the layer thickness, are introduced. The two small parameters arisen from this approach are determined by, respectively, the period of the coefficients of the pertinent equations and the layer thickness, which may or may not be of the same order of magnitude. The analytical formulae for effective moduli derived herein make it possible to gain useful insight into the manner in which the geometrical and mechanical properties of the individual constituents affect the elastic properties of the thin geometrically nonlinear composite layer with wavy surfaces.

Keywords: Composite structures; Nonlinearity; Asymptotic homogenization; Multi-scale modeling; Elastic coefficients

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Introduction

The preponderance of uses for composite materials is in the form of plates and shells, the optimum strength-to-weight characteristics of which offer engineers attractive alternatives for different applications. A large fraction of these applications is in the structural, aerospace, and marine fields, where the composites are made of continuous fibers in polymeric matrix to obtain fiber-reinforced polymer laminated composite plates or shells. The geometry of such composite structures is governed by periodic configuration, i.e. reinforcements are regularly distributed with very significant coordinate effects, so as to reap the benefit of carrying smaller weights under certain loading conditions. However, the practical issue in the mechanics of advanced composites is the determination of the effective properties of these structures which will naturally be dependent on the spatial distribution of fibers, geometric characteristics, and the mechanical properties of the constituents involved.

The micromechanical analysis of regular periodic composite structures made up of reinforcements in a matrix has been the focus of investigation for long time. The first general solution to the equations of linear elasticity corresponding to thin plates by use of the method of series expansion was presented by Cauchy in early 1800s [1]. The 'Classical Laminated Plate Theory' is based on the assumption that normal to mid-plane before deformation remains straight and normal to the plane after deformation, and the effects of transverse shear strains were ignored. Consequently, the applicability of the general solution has been confined to plates strictly with limited thickness subjected to edge tractions through the use of a series of biharmonic functions [2]. In the Hencky-Mindlin theories, the displacements are expanded in powers of the thickness of the plate [3]. A geometrically nonlinear theory associated with the classical plate theory was considered by Reissner [4]. Levinson expanded such a plate theory by considering in-plane displacements in terms of cubic functions of the thickness coordinate [5]. Unfortunately, the theory is developed based on a variationally inconsistent set of equilibrium equations and therefore did not correctly account for all of the strain energy associated with the displacement field.

The continued interest in finding the exact solution for elastostatic problem consisting of thick laminates made up of orthotropic layers has brought the attention to asymptotic expansion and its application into the analysis of plates with periodic composite structures. In the initial elastostatic problem of a plate with composite periodic structure, two small parameters, namely the plate thickness h and an in-plane dimension ε of the periodicity cell, were considered. In the classical

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homogenization theory, plates of constant thickness were considered with simultaneous reduction of all dimensions of the periodicity cells by Caillerie [6]. Homogeneous plates of rapidly varying thickness have been studied independently by Kohn and Vogelius [7-9] with the help of methods similar to those used by Caillerie [10]. Kalamkarov [11] has generalized the above two approaches to the case of shells with rapidly varying material properties and thickness. In the work by Saha *et al.* [12], the general model was expanded to the application of smart composite shells with periodically arranged actuators and varying thickness using the asymptotic homogenization technique. The current paper is aimed at developing higher order terms of the asymptotic expansions that model the deformations of thin composite layers of wavy surfaces. To the ultimate objective, the modified $\varepsilon = h\delta$ homogenization method is applied to the study of a curved thin composite plate with a regular structure in the context of linear and geometrically nonlinear elasticity theory. Here, the starting point is the exact three-dimensional formulation of the problem, without resource to the Kirchhoff-Love hypothesis. Owing the small parameter δ, then three-dimensional problem is proved to be amenable to a rigorous asymptotic analysis unifying an asymptotic three- to two-dimensions process and a homogenous material process.

Problem Formulation

We apply the geometrically nonlinear theory of elasticity to a thin periodically nonhomogeneous (composite material) layer with wavy surfaces. We consider a triorthogonal dimensionless coordinate system $\alpha_1, \alpha_2, \alpha_3$ such that the coordinate lines α_1 and α_2 agree with the lines of principal curvature of the middle surface (for $\alpha_3 = 0$), while the α_3 axis is directed along its normal. It follows then that the tangent vectors at the reference surface are correlated by:

$$\boldsymbol{a}_i(\alpha_1, \alpha_2) \equiv \boldsymbol{g}_i(\alpha_1, \alpha_2, 0)$$

Accordingly, the reciprocal base vectors $\boldsymbol{a}_i$ are normal to the $\boldsymbol{\alpha}^i$ surface according to:

$$\boldsymbol{a}^i(\alpha^1, \alpha^2) \equiv \boldsymbol{g}^i(\alpha^1, \alpha^2, 0); \text{where } \boldsymbol{a}^i . \boldsymbol{a}_j = \delta^i_j$$

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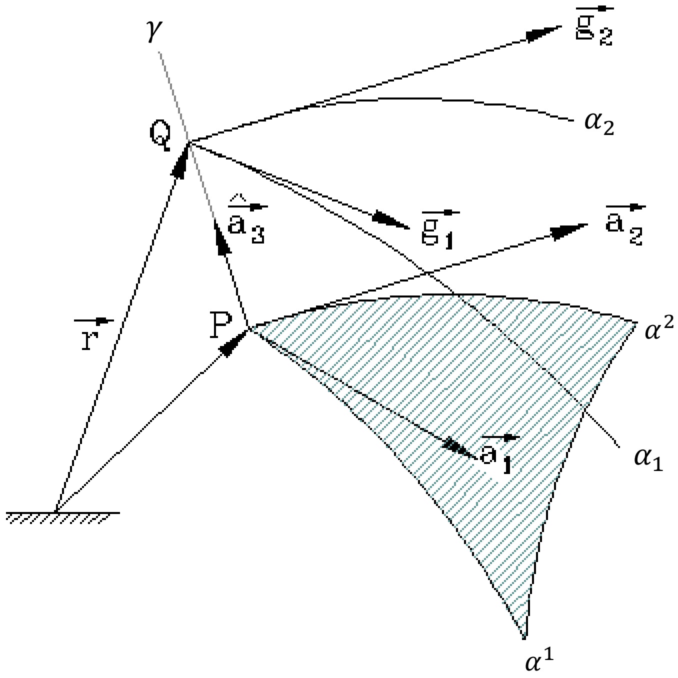

Fig. 1: Position and coordinates of a surface representing the middle surface of an arbitrary shell element.

Now consider an undeformed shell layer representing an inhomogeneous solid occupying domain Ω_δ with boundary $\partial\Omega_\delta$, as shown in Fig. 2.

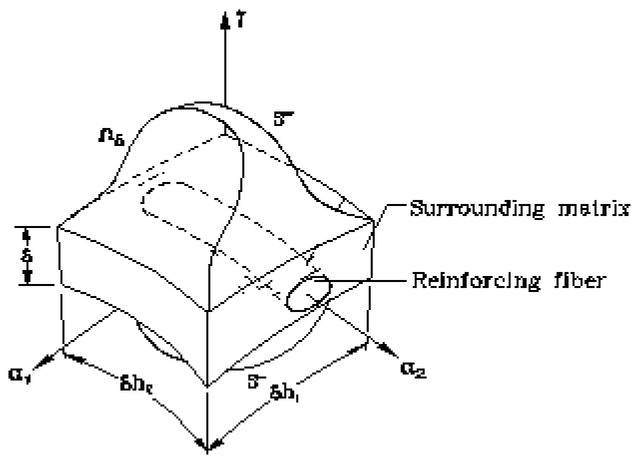

Fig. 2: Non-homogeneous composite shell with arbitrary surfaces.

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Such a coordinate system can be represented by the Lamé coefficients:

$$H_1 = A_1(1 + k_1\gamma), \qquad H_2 = A_2(1 + k_2\gamma) \tag{1}$$

where $A_1(\alpha_1, \alpha_2)$ and $A_2(\alpha_1, \alpha_2)$ are the Lamé parameters of the middle surface, and $k_1(\alpha_1, \alpha_2)$ and $k_2(\alpha_1, \alpha_2)$ are the principal curvatures of the middle surface. The unit cell of the problem can be defined by:

$$\left\{-\frac{\delta h_1}{2} < \alpha_1 < \frac{\delta h_1}{2}, \qquad -\frac{\delta h_2}{2} < \alpha_2 < \frac{\delta h_2}{2}, \qquad x_3^- < x_3 < x_3^+\right\} \tag{2}$$

where $x_3^{\pm} = \pm\frac{\delta}{2} \pm \delta F^{\pm}\left(\frac{x_1}{\delta h_1}, \ \frac{x_2}{\delta h_2}\right)$, $\delta \ll 1$.

Small parameter δ determines the thickness of the unit cell and h_1 and h_2 are the ratio of the corresponding arc length of the middle surface along α_1 and α_2 directions to the thickness (dimensionless) of the unit cell. Functions $F^{\pm}$ allow for the variation in top and bottom surfaces. Since they are normalized with respect to dimensionless thickness of the unit cell, i.e. $F = \frac{\Phi}{\delta}$, they are 1-periodic in corresponding fast variables $\frac{\alpha_1}{\delta h_1}$ and $\frac{\alpha_2}{\delta h_2}$ as they span the entire length of the unit cell.

The equilibrium equations can be written in the following form:

$$\begin{gathered}
\frac{\partial(H_2\sigma_{11})}{\partial\alpha_1} + \frac{\partial(H_1\sigma_{12})}{\partial\alpha_2} + \frac{\partial(H_1H_2\sigma_{13})}{\partial\gamma} - \frac{H_1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\sigma_{22} + \frac{H_2}{A_2}\frac{\partial A_1}{\partial\alpha_2}\sigma_{21} + H_2A_1k_1\sigma_{31} \\
+ H_1H_2p_1 = 0 \\
\frac{\partial(H_2\sigma_{21})}{\partial\alpha_1} + \frac{\partial(H_1\sigma_{22})}{\partial\alpha_2} + \frac{\partial(H_1H_2\sigma_{23})}{\partial\gamma} - \frac{H_2}{A_2}\frac{\partial A_1}{\partial\alpha_2}\sigma_{11} + \frac{H_1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\sigma_{12} + H_1A_2k_2\sigma_{32} \\
+ H_1H_2p_2 = 0 \\
\frac{\partial(H_2\sigma_{31})}{\partial\alpha_1} + \frac{\partial(H_1\sigma_{32})}{\partial\alpha_2} + \frac{\partial(H_1H_2\sigma_{33})}{\partial\gamma} - H_2A_1k_1\sigma_{11} - H_1A_2k_2\sigma_{22} + H_1H_2p_3 = 0
\end{gathered} \tag{3}$$

where p_i $(i = 1,2,3)$ are the body force components.

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The physical components of the strain tensor e_{kl} and the stress tensor σ_{ij} are connected by:

$$\sigma_{ij} = C_{ijkl}e_{kl} \tag{4}$$

where C_{ijkl} are the coefficients of elasticity. Here and henceforth, summation is over identical subscripts where $i, j, k, l = 1, 2, 3$. It is assumed that elongations and strains are small and the nonlinear equations of motion are therefore of the form:

$$t_{ij,i} + P_j^* = 0 \tag{5}$$

$$t_{ij} = \sigma_{ij} + \sigma_{li}u_{j,l} \tag{6}$$

where $\frac{\partial u_j}{\partial x_l} = u_{j,l}$.

We assume that the surfaces of the shell are subjected to forces $\sigma_{ij}n_j^{\pm} = p_i^{\pm}$, where $n_j^{\pm}$ correspond to the unit normals to the surfaces $S^{\pm}$, and $p_i^{\pm}$ are the contravariant load components.

$$e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} + \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_l}\right) \tag{7}$$

$$\begin{aligned} => 2e_{kl} = \Bigg(&\frac{\partial u_k}{\partial x_l} + \frac{1}{\delta h_l}\frac{\partial u_k}{\partial y_l} + \frac{\partial u_l}{\partial x_k} + \frac{1}{\delta h_k}\frac{\partial u_l}{\partial y_k} + \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_l} + \frac{1}{\delta h_l}\frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial y_l} \\ &+ \frac{1}{\delta h_k}\frac{\partial u_m}{\partial y_k}\frac{\partial u_m}{\partial x_l} + \frac{1}{\delta^2 h_k h_l}\frac{\partial u_m}{\partial y_k}\frac{\partial u_m}{\partial y_l}\Bigg) \end{aligned} \tag{8}$$

We introduce the 'rapid' coefficients $y_1 = x_1/(\delta h_1)\,, y_2 = x_2/(\delta h_2)\,,$ and $z = x_3/\delta$ to distinguish between 'rapid' and 'slow' variables when performing differentiation. The solution of the problem is represented as an asymptotic series expansion in powers of the small parameter in the form:

$$u_i = u_i^{(0)}(\vec{x}) + \delta u_i^{(1)}(\vec{x},\vec{y},z) + \delta^2 u_i^{(2)}(\vec{x},\vec{y},z) + \cdots \tag{9}$$

where $\boldsymbol{x} = (x_1, x_2), \boldsymbol{y} = (y_1, y_2)$ and the functions $u_i^{(l)}(\vec{x},\vec{y},z)$ for $l = 1,2, \ldots$ are 1-periodicin in y_1 and y_2.

By applying the asymptotic expansion to the external forces we may write:

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$$P_\nu^* = \delta f_\nu^*(\vec{x},\vec{y},z), \qquad P_3^* = \delta^2 f_3^*(\vec{x},\vec{y},z) \tag{10}$$
$$p_\nu^{*\pm} = \delta^2 g_\nu^{*\pm}(\vec{x},\vec{y}), \qquad p_3^{*\pm} = \delta^3 g_3^{*\pm}(\vec{x},\vec{y}) \qquad (y = 1,2)$$

where all functions involved are periodic in y_1 and y_2, with the unit cell Ω defined by:

$$\left\{y_1, y_2 \in \left(-\frac{1}{2},\frac{1}{2}\right), \qquad z \in (z^-, z^+)\right\}, \qquad z^\pm = \pm\frac{1}{2} \pm F^\pm(\boldsymbol{y}) \tag{11}$$

Likewise, the periodicity property is applied in the elastic coefficients $C_{ijkl}(\boldsymbol{y}, z)$, which are visualized as piecewise-smooth functions undergoing discontinuities of the first-kind at the (non-intersecting) contact surfaces between the dissimilar phases of the composite.

It then follows from Equations (6), (7), and (9) that

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \delta\sigma_{ij}^{(1)} + \delta^2\sigma_{ij}^{(2)} + \cdots \tag{12}$$
$$t_{ij} = t_{ij}^{(0)} + \delta t_{ij}^{(1)} + \delta^2 t_{ij}^{(2)} + \cdots$$

Using these and Equation (10) in (5) yields the following δ-expressions:

$$\frac{1}{\delta}H_j^{(-1)} + H_j^{(0)} + \delta H_j^{(1)} + \delta^2 H_j^{(2)} + \cdots = 0$$
$$H_j^{(-1)} = t_{3j,3}^{(0)} + \frac{1}{h_\alpha}t_{\alpha j,\alpha}^{(0)}$$
$$H_j^{(0)} = t_{\alpha j,\alpha}^{(0)} + t_{3j,3}^{(1)} + \frac{1}{h_\alpha}t_{\alpha j,\alpha}^{(1)} \tag{13}$$
$$H_j^{(1)} = t_{\alpha j,\alpha}^{(1)} + t_{3j,3}^{(2)} + \frac{1}{h_\alpha}t_{\alpha j,\alpha}^{(2)} + f_j^*\left(\delta_{j1} + \delta_{j2}\right)$$
$$H_j^{(2)} = t_{\alpha j,\alpha}^{(2)} + t_{3j,3}^{(3)} + \frac{1}{h_\alpha}t_{\alpha j,\alpha}^{(3)} + f_j^*\delta_{j3}$$

$$\left(t_{ij}^{(0)} + \delta t_{ij}^{(1)} + \delta^2 t_{ij}^{(2)} + \delta^3 t_{ij}^{(3)} + \cdots\right)n_i^\pm = \delta^2 g_j^{*\pm}\left(\delta_{j1} + \delta_{j2}\right) \pm \delta^3 g_j^{*\pm}\delta_{j3}, \quad (z = z^\pm) \tag{14}$$

where, as before, the range of Greek indices is 1 and 2, while Latin indices take on 1, 2, and 3. We denote

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$$\boldsymbol{L}_{ijn} = c_{ijn\mu}\frac{1}{h_\mu}\frac{\delta}{\delta y_\mu} + c_{ijn3}\frac{\delta}{\delta z} \tag{15}$$

The leading terms in Equation (12) may then be written as

$$\begin{aligned}\sigma_{ij}^{(0)} &= \boldsymbol{L}_{ijk}u_k^{(1)} + c_{ijk\alpha}u_{k,\alpha}^{(0)} + \frac{1}{2h_\mu}u_{m|\mu}^{(1)}\boldsymbol{L}_{ij\mu}u_m^{(1)} + \frac{1}{2}u_{m|3}^{(1)}\boldsymbol{L}_{ij3}u_m^{(1)} + u_{m,\alpha}^{(0)}\boldsymbol{L}_{ij\alpha}u_m^{(1)} \\ &\quad + \frac{1}{2}c_{ij\alpha\beta}u_{m,\alpha}^{(0)}u_{m,\beta}^{(0)} \\ t_{ij}^{(0)} &= \sigma_{ij}^{(0)} + \sigma_{i\beta}^{(0)}u_{j,\beta}^{(0)} + \sigma_{i3}^{(0)}u_{j|3}^{(1)} + \frac{1}{h_\beta}\sigma_{i\beta}^{(0)}u_{j|\beta}^{(1)}\end{aligned} \tag{16}$$

The problem of determining $t_{ij}^{(0)}$ follows from Equations (13) and (14) as

$$H_j^{(-1)} = 0, \qquad t_{ij}^{(0)}n_j^{\pm} = 0 \qquad (z = z^{\pm}) \tag{17}$$

The substitution of Equation (16) yields a problem for the functions $u_k^{(1)}$, in which we shall ignore the terms containing products of three or more derivatives of displacement components with respect to the 'slow' coordinates x_α, where $\alpha = 1{,}2$.

The solution of the problem in Equations (16) and (17) may be represented in the form:

$$u_k^{(1)} = v_k^{(1)}(\boldsymbol{x}) + U_k^{n\mu}(\boldsymbol{y},z)u_{n,\mu}^{(0)} + W_k^{mn\lambda\mu}(\boldsymbol{y},z)u_{m,\mu}^{(0)}u_{n,\mu}^{(0)} \tag{18}$$

with the provision that the functions $U_k^{n\mu}(\boldsymbol{y},z)$ and $W_k^{mn\lambda\mu}(\boldsymbol{y},z)$ are 1-periodic in y_1 and y_2 and solve the following local problems:

$$\begin{aligned}&\frac{1}{h_\beta}b_{i3|3}^{n\mu} = 0, \qquad b_{ij}^{n\mu} = \boldsymbol{L}_{ijk}U_k^{n\mu} + c_{ijn\mu} \\ &b_{ij}^{n\mu}n_j^{\pm} = 0 \qquad (z = z^{\pm})\end{aligned} \tag{19}$$

$$\begin{aligned}&\frac{1}{h_\beta}\left(B_{i\beta}^{mn\lambda\mu} + b_{\alpha\beta}^{m\lambda}\frac{1}{h_\alpha}U_{i|\alpha}^{n\mu} + b_{3\beta}^{m\lambda}U_{i|3}^{n\mu}\right)_{|\beta} + \left(B_{i3}^{mn\lambda\mu} + b_{\alpha3}^{m\lambda}\frac{1}{h_\alpha}U_{i|\alpha}^{n\mu} + b_{33}^{m\lambda}U_{i|3}^{n\mu}\right)_{|3} = 0 \\ &B_{ij}^{mn\lambda\mu}n_j^{\pm} = 0 \qquad (z = z^{\pm})\end{aligned} \tag{20}$$

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$$B_{ij}^{mn\lambda\mu} = \boldsymbol{L}_{ijk} W_k^{mn\lambda\mu} + \frac{1}{2h_\alpha} U_{k|\alpha}^{m\lambda} \boldsymbol{L}_{ij\alpha} U_k^{n\mu} + \frac{1}{2} U_{k|3}^{m\lambda} \boldsymbol{L}_{ij3} U_k^{n\mu} + \boldsymbol{L}_{ij\lambda} U_m^{n\mu} + \frac{1}{2} c_{ij\lambda\mu} \delta_{mn} \quad (21)$$

It is seen that problem in Equation (19) coincides with local problem for a thin shell in the framework of linear elasticity theory.

Note that at the surfaces where discontinuities in material properties occur, continuity conditions must be added to the above problems in Equations (19) and (20).

Now, for $n\mu = 31, 32$, it can be shown that problem (19) has an exact solution, given by equations

$$U_1^{31} = -z, \quad U_2^{31} = U_3^{31} = 0, \quad U_2^{32} = -z, \quad U_1^{32} = U_3^{32} = 0 \quad (22)$$

and as a result,

$$b_{ij}^{3\mu} = 0 \quad (23)$$

Substituting above relations into Equation (20) reduces it to a much simpler form for $mn = 33$, as follows:

$$\begin{gathered} \frac{1}{h_\beta} B_{i\beta|\beta}^{33\lambda\mu} + B_{i3|3}^{33\lambda\mu} = 0 \\ B_{ij}^{33\lambda\mu} n_j^{\pm} = 0 \qquad (z = z^{\pm}) \\ B_{ij}^{33\lambda\mu} = \boldsymbol{L}_{ijk} W_k^{33\lambda\mu} + \frac{1}{2} c_{ij33} \delta_{\lambda\mu} + \frac{1}{2} c_{ij\lambda\mu} \end{gathered} \quad (24)$$

Comparing the local problems (19) for the functions $U_k^{\lambda\mu}$ and (24) for $W_k^{33\lambda\mu}$ it can be shown that

$$\begin{gathered} W_\alpha^{33\lambda\mu} = \frac{1}{2} U_\alpha^{\lambda\mu} \qquad (\alpha = 1,2) \\ W_3^{33\lambda\mu} = \frac{1}{2}\left(U_3^{\lambda\mu} - z\delta_{\lambda\mu}\right) \end{gathered} \quad (25)$$

and using this in Equation (24) for $B_{ij}^{33\lambda\mu}$ yields

$$B_{ij}^{33\lambda\mu} = \frac{1}{2} b_{ij}^{\lambda\mu} \quad (26)$$

after comparing with Equation (19) for $b_{ij}^{\lambda\mu}$.

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In what follows, we substitute Equations (18) and (16) and use the notation of Equation (19) and (20) to obtain:

$$\sigma_{ij}^{(0)} = b_{ij}^{\lambda\mu} u_{\lambda,\mu}^{(0)} + B_{ij}^{mn\lambda\mu} u_{m,\lambda}^{(0)} u_{n,\mu}^{(0)} \tag{27}$$

Application of the method of homogenization and the use of Equation (17) now yields the problem from which the leading order terms in Equations (13) and (17) can be determined:

$$H_j^{(0)} = \langle H_j^{(0)} \rangle, \qquad t_{ij}^{(1)} n_i^{\pm} = 0 \quad (z = z^{\pm}) \tag{28}$$

where the volume average is defined as follows:

$$\langle \varphi \rangle = \frac{1}{|\Omega|} \int_{\Omega} \varphi \, dy_1 dy_2 dz \tag{29}$$

Using the periodicity in y_1 and y_2 and conditions (28) at $z = z^{\pm}$ it is found from Equation (13) that

$$\langle H_j^{(0)} \rangle = t_{\alpha j}^{(0)}{}_{,\alpha} \tag{30}$$

Following the earlier found solution of the linear version of the problem (28) and (30), we write

$$u_1^{(0)} = u_2^{(0)} = 0, \qquad u_3^{(0)} = w(x), \qquad v_3^{(1)}(\boldsymbol{x}) = 0 \tag{31}$$

This gives

$$\begin{gathered} u_\alpha^{(1)} = v_\alpha^{(1)}(\boldsymbol{x}) - z w_{,\alpha} + \frac{1}{2} U_\alpha^{\lambda\mu} w_{,\lambda} w_{,\mu} \\ u_3^{(1)} = \frac{1}{2}\left(U_3^{\lambda\mu} - z\delta_{\lambda\mu}\right) w_{,\lambda} w_{,\mu} \end{gathered} \tag{32}$$

using Equations (18) and (25), and hence by Equations (27) and (26):

$$\sigma_{ij}^{(0)} = \frac{1}{2} b_{ij}^{\lambda\mu} w_{,\lambda} w_{,\mu} \tag{33}$$

Using Equations (6), (7), (31), and (32) along with (4) we find, within the accuracy of the calculation:

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$$\sigma_{ij}^{(1)} = \boldsymbol{L}_{ijk}u_k^{(2)} + c_{ij\alpha\beta}\varepsilon_{\alpha\beta}^{(1)} + zc_{ij\alpha\beta}\tau_{\alpha\beta} + w_{,\alpha}\left(\boldsymbol{L}_{ij\alpha}u_3^{(2)} - \boldsymbol{L}_{ij3}u_\alpha^{(2)}\right) - c_{ij3\beta}w_{,\alpha}\varepsilon_{\alpha\beta}^{(1)} - c_{ijm\beta}U_m^{\alpha\mu}w_{,\alpha}\tau_{\mu\beta}$$
$$t_{ij}^{(1)} = \sigma_{ij}^{(1)} + \sigma_{i\beta}^{(1)}w_{,\beta}\delta_{j3} - \sigma_{i3}^{(1)}w_{,\beta}\delta_{j\beta} \quad (34)$$

denoting,

$$\varepsilon_{\alpha\beta}^{(1)} = v_{\alpha,\beta}^{(1)}, \qquad \tau_{\alpha\beta} = -w_{,\alpha\beta} \quad (35)$$

Now if we substitute Equations (30) and (34)-(35) into (28) and make use of (16) and (33), a problem for determining the functions $u_k^{(2)}$ will be obtained, the solution of which may be represented to the same accuracy in the following form:

$$u_k^{(2)} = U_k^{\lambda\mu}\varepsilon_{\lambda\mu}^{(1)} + V_k^{\lambda\mu}\tau_{\lambda\mu} + Q_k^{\alpha\lambda\mu}w_{,\alpha}\varepsilon_{\lambda\mu}^{(1)} + R_k^{\alpha\lambda\mu}w_{,\alpha}\tau_{\lambda\mu} \quad (36)$$

Here the functions $U_k^{\lambda\mu}(\boldsymbol{y},z), V_k^{\lambda\mu}(\boldsymbol{y},z), Q_k^{\alpha\lambda\mu}(\boldsymbol{y},z)$ and $R_k^{\alpha\lambda\mu}(\boldsymbol{y},z)$ are 1-periodic in y_1 and y_2 with the unit cell Ω and solve the following local problems:

$$\frac{1}{h_\beta}b_{i\beta|\beta}^{\lambda\mu} + b_{i3|3}^{\lambda\mu} = 0, \quad b_{ij}^{\lambda\mu}n_j^{\pm} = 0, \quad (z = z^{\pm}), \qquad \left(b_{ij}^{\lambda\mu} \leftrightarrow b_{ij}^{*\lambda\mu} \leftrightarrow q_{ij}^{\alpha\lambda\mu}\right), \quad (37)$$

$$\frac{1}{h_\beta}r_{i\beta|\beta}^{\alpha\lambda\mu} + r_{i3|3}^{\alpha\lambda\mu} = b_{i\mu}^{\lambda\alpha} - \langle b_{i\mu}^{\lambda\alpha}\rangle, \quad r_{ij}^{\alpha\lambda\mu}n_j^{\pm} = 0, \quad (z = z^{\pm}), \quad (38)$$

$$\begin{aligned} b_{ij}^{\lambda\mu} &= \boldsymbol{L}_{ijk}U_k^{\lambda\mu} + c_{ij\lambda\mu}, \quad b_{ij}^{*\lambda\mu} = \boldsymbol{L}_{ijk}V_k^{\lambda\mu} + zc_{ij\lambda\mu}, \\ q_{ij}^{\alpha\lambda\mu} &= \boldsymbol{L}_{ijk}Q_k^{\alpha\lambda\mu} + \boldsymbol{L}_{ij\alpha}U_3^{\lambda\mu} - \boldsymbol{L}_{ij3}U_\alpha^{\lambda\mu} - c_{ij3\mu}\delta_{\alpha\lambda}, \\ r_{ij}^{\alpha\lambda\mu} &= \boldsymbol{L}_{ijk}R_k^{\alpha\lambda\mu} + \boldsymbol{L}_{ij\alpha}V_3^{\lambda\mu} - \boldsymbol{L}_{ij3}V_\alpha^{\lambda\mu} - c_{ijm\mu}U_m^{\alpha\lambda} \end{aligned} \quad (39)$$

To the foregoing equations we have to adjoin the jump conditions at the material surfaces of discontinuity, which we give in the following form:

$$\left[\!\left[U_k^{\lambda\mu}\right]\!\right] = 0 \qquad \left(U_k^{\lambda\mu} \leftrightarrow V_k^{\lambda\mu} \leftrightarrow Q_k^{\alpha\lambda\mu} \leftrightarrow R_k^{\alpha\lambda\mu}\right),$$
$$\left[\!\left[\frac{1}{h_\beta}n_\beta^{(k)}b_{i\beta}^{\lambda\mu} + n_3^{(k)}b_{i3}^{\lambda\mu}\right]\!\right] = 0 \qquad \left(b_{ij}^{\lambda\mu} \leftrightarrow b_{ij}^{*\lambda\mu} \leftrightarrow q_{ij}^{\alpha\lambda\mu} \leftrightarrow r_{ij}^{\alpha\lambda\mu}\right) \quad (40)$$

where $n_i^{(k)}$ denotes the unit normal at the surface of discontinuity, related to the coordinate system y_1, y_2, z. This is in contrast to the $z = z^{\pm}$ conditions in the local problems (19), (20), and (24), and (37) and (38), where $n_i^{\pm}$, the unit normal to the surfaces $S^{\pm}$, are related to the coordinate system

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x_1, x_2, x_3. In actually solving these problems, however, it proves more convenient to rewrite them using the unit normals $n_i^{\pm(y)}$ related to y_1, y_2, z.

The local problems in Equation (40) are linear in the unknown functions, and their solutions are unique up to constant terms. This ambiguity is removed by imposing conditions in a form:

$$\langle U_k^{\lambda\mu} \rangle = 0 \quad \text{when} \quad z = 0 \qquad \left(U_k^{\lambda\mu} \leftrightarrow V_k^{\lambda\mu} \leftrightarrow Q_k^{\alpha\lambda\mu} \leftrightarrow R_k^{\alpha\lambda\mu}\right), \tag{41}$$

where $\langle \dots \rangle_y$ indicates average with respect to y_1 and y_2 only.

Note that the problems for the functions $U_k^{\lambda\mu}$ and $V_k^{\lambda\mu}$ are identical in form to the corresponding local problems of the linear theory of elasticity, and in the remaining two problems $Q_k^{\alpha\lambda\mu}$ and $R_k^{\alpha\lambda\mu}$ these functions are considered to be known.

Substituting Equation (36) into (34) and using the notation of (39) we arrive at

$$\sigma_{ij}^{(1)} = b_{ij}^{\lambda\mu} \varepsilon_{\lambda\mu}^{(1)} + b_{ij}^{*\lambda\mu} \tau_{\lambda\mu} + q_{ij}^{\alpha\lambda\mu} w_{,\alpha} \varepsilon_{\lambda\mu}^{(1)} + r_{ij}^{\alpha\lambda\mu} w_{,\alpha} \tau_{\lambda\mu} \tag{42}$$

Returning now to Equations (37) and (38), we average their left-hand sides after first multiplying them by z and z^2 and we take into account, in doing so, the $z = z^{\pm}$ boundary conditions and periodicity in y_1 and y_2. This gives the following relations for the effective elastic moduli of the homogenized shell in the form:

$$\langle b_{i3}^{\lambda\mu} \rangle = \langle z b_{i3}^{\lambda\mu} \rangle = 0, \qquad b_{i3}^{\lambda\mu} \leftrightarrow b_{i3}^{*\lambda\mu} \leftrightarrow q_{i3}^{\alpha\lambda\mu} \tag{43}$$
$$\langle r_{i3}^{\alpha\lambda\mu} \rangle = \langle z \rangle \langle b_{i\mu}^{\alpha\lambda} \rangle - \langle z b_{i\mu}^{\alpha\lambda} \rangle$$

The following symmetry properties hold:

$$b_{ij}^{mn} = b_{ji}^{mn} = b_{ij}^{nm} \qquad \left(b_{ij}^{mn} \leftrightarrow b_{ij}^{*mn}\right) \tag{44}$$

Finally, from Equations (16) and (34):

$$\langle t_{\alpha\beta}^{(0)} \rangle = \langle \sigma_{\alpha\beta}^{(0)} \rangle, \qquad \langle t_{\alpha3}^{(0)} \rangle = \langle \sigma_{\alpha\beta}^{(0)} \rangle w_{,\beta}$$
$$\langle t_{\alpha\beta}^{(1)} \rangle = \langle \sigma_{\alpha\beta}^{(1)} \rangle, \qquad \langle z t_{\alpha\beta}^{(1)} \rangle = \langle z \sigma_{\alpha\beta}^{(1)} \rangle \tag{45}$$
$$\langle t_{\alpha3}^{(1)} \rangle = \langle \sigma_{\alpha\beta}^{(1)} \rangle w_{,\beta} + \langle r_{\alpha3}^{\beta\lambda\mu} \rangle w_{,\beta} \tau_{\lambda\mu}$$

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where Equations (31)-(33), (42) and (43) have also been used.

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